Memoir.

Science : Mathematics.

written by

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Subject :

Rational points on general varieties.

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□

To all of those I owe the essential and almost everything. Family and acquaintances; recent or from a long time; closed or far away; they were those who make me understand that this essential was not mathematics. Hopefully, they will understand my deviating from their advice.

Acknowledgements.[1]

I would like to thank everyone I have come across with for their pleasant company or support, weither they are part of the scientific world or not, a thought to the variegated mathematical community of Jussieu-Chevaleret from the Mathematical Institute of the same name, whose daily teatime initiative is noticeable. A special thought for the members of the Mathematics Department of Denis University Diderot and in particular; the Professors Paul Gerardin, Michael Harris, Marc Hindry, Bernard Maurey and Jean-Francois Mestre; to Patrice Perrin who confirmed me on the rails of arithmetics, a long time ago. I also send a few thoughts to the mathematicians of « Théorie des Nombres » project of the Mathematical « Institute », especially to Professor Michel Waldschmidt, as well as to (pre and post) phd students with whom I have discussed mathematics or other things, too numerous to be cited without omission; finally to the team of the Mathematics-Research Library as well as the administrative team of the Mathematics UFR of Denis Diderot University and more generally of the Institute and especially, last but not least, to those often anonymous, who maintain this place and make it what it is.

This electronic version of this memoir was produced entirely from free software. This is an opportunity to thank the Free Software communities, particularly those of users-developers of TeX/LaTeX and the software listed below; as well as all those, mathematicians, computer scientists or not, who put their resources and work online. The operating system used was the latest stable version of the Debian/GNU Linux distribution, known as « Sarge» which I recommend for its modularity and completeness, making it possible to run on obsolete machines with low resources. The LaTeX sources were written with the little GNU /nano editor, customized with a lively syntax highlighting. All illustrations are coded in scripts from the LaTeX Mfpic package, apart from the three-dimensional ones coded with scripts of the PovRay software. Some computations and verifications were performed using the Pari/GP software suite dedicated to number theory.

[1] This one is from the french version of 2005 : the new version one is in the appendix.

Summary. The subject of this short memoir is the study of the existence of rational points on general varieties, not necessarily algebraic. We briefly mention the different approaches to this problem and we present a conjecture that seems absent in the literature.

Keywords. Hypersurfaces, rational points, arithmetic geometry, plane curves, Hasse Principle, Brauer-Manin obstruction, algebraic independence, periods.

MSC (2000) : 11DXX, 11GXX, 14G05, 14H50, 14J10, 14J70.

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Introduction, summary and notations.

0.1 Introduction

The Arithmetic investigations on varieties have a very long history of at least three thousand years for algebraic varieties with the study of diophantine problems during Babylonian times and have given birth only a few decades ago to two recent branches of Number Theory; namely Arithmetic-Geometry and Diophantine-Geometry; that can be aggregated in the another new branch of Arithmetic-Diophantine Geometry.

More generally, since its birth with the study of diophantine problems of the earliest Babylonian tablets found so far; Number Theory has been stimulating numerous generations of mathematicians till nowadays; where it founds a very rich array of applications in modern computer and data sciences especially within the cryptographic basement of the new digital economy.

Another reason for that appeal, besides deep ontological ones, like the intrinsic nature of integers; is that, Number Theory has not answered to questions raised by hundreds of conjectures; some having quite simple formulations accessible to a wide public audience; this is, to my opinion, why one of the greatest mathematicians of all times, Carl Friedrich Gauß qualified Number Theory as the Queen of Mathematics.

Hence we can say that Number theory started with Diophantine problems of the Babylonian era, those being the study of integer solutions of algebraic equations defined over the ring of integers; they are encompassed by Arithmetic that is the study of rational points of algebraic equations defined over the field of rational numbers. It is worthy to note on the go, as an anachronism fixing remark, that Diophante only comes into play quite a long time after the Babylonian first traces of mathematics engraved in tablets, something like 1500 years after; but the problems tackled by Babylonian scribes were the same as the ones that were studied by Diophante and its contemporaries : this is Mathematics, a perpetual requestioning and rephrasing of the same eternal problematics.

As motivations for studying rational points on varieties, if ever needed, we can enumerate the following points.

1. History : such a long term interest of mankind towards Arithmetic problems suffices to legitimate the continuation of their study.

2. In a more pragmatic perspective : almost all the security layer of the Modern Digital Economy transactions is based on Arithmetic.

3. Curiosity : exploring less known paths in answering arithmetic questions.

Although successive generations of arithmeticians have answered finiteness questions about rational points for particular classes of algebraic varieties; the mathematicians community still has a lot to do about the problem of the existence of such points. Those questions about rational points on varieties, as shown in the introductory diagram 1 p 6, spread out within several theories; hardly circumscribing the whole fields of possible investigations.

As we mentioned before, for the case of algebraic varieties; an heavy machinery elaborated during the long term development of the related theories is available; in the form of an arsenal of techniques steadily developed during millennia of slow maturation that allows us to tackle these problems; since the theory, dating back at least to ancient Babylonian times, is still being developed actively nowadays : there are actually few areas of mathematics having such a long term evolution so that it is tempting to affirm that Mathematics began with Number Theory and one of its main branches that is Arithmetic.

For the non-algebraic varieties case, finiteness of rational points issues are related to Transcendental Number Theory and particularly to the nature of values taken by transcendental functions at algebraic points.

When it is known that the studied varieties have lots of rational points; the theory focuses on their distribution; often involving another two of its numerous branches; namely "Analytic Number Theory" and the "Geometry of Numbers", both interconnected by the methods of sieves, lattices and calculus approximations as illustrated, for example, in the work of Jonathan Pila [Pil91].

After this short introduction, let's come back to the contents of this brief memoir.

Summary

In the first part, we recall roughly what is known for algebraic varieties, we will try to give a brief overview of recent advances in the theory without any pretention for exhaustivity nor even less for originality. This part content is essentially the same as the first french version modulo some corrections and addings.

In a second part, we propose and study a conjecture tackling both cases of algebraic and non-algebraic varieties; extending widely the investigations to differentiable or piecewise smooth varieties, modulo a restriction on their size : the underlying goal of this memoir is the study of a possible extension of a criterion for the presence of rational points on varieties, to a much larger class of varieties than the traditionally studied algebraic ones. This second part content has been nearly fully rewritten during the translation process after a partial recompilation of the author past research paper notes.

The hasty number theorist researcher may skip all the first chapter and go directly to the second one and its synthesis sections near the end of the memoir; to harvest and pick-up; hopefully some potentially interesting ideas at "the cutting-edge" of the present knowledge; then relax by reading the next hectic biographical appendix; that finally gives a "circonstanciée" human tint to the conclusion of that memoir.

The prerequisites for reading this text are fairly modest : a solid ending undergraduate or a beginning graduate level should be largely enough; but decent efforts have been made to broaden its audience and make it readable by the enthusiastic undergraduate students, teasing their curiosity with the secret aim of passing them the relay. Hopefully; anyone coming across that text will have something worthy to grab in it : this was the main purpose of that translated version.

0.2 Notations and conventions.

Now, a few foundational words : the mathematical setting considered in the previous lines, and that will be considered in the sequel of this memoir, is the Standard one of Modern Mathematics, based on the Zermelo-Fraenkel logic model of Set theory with the Axiom of Choice, often abreviated (ZFC).

This model assumes the existence of an infinite set as an axiom : the reader should bear in mind that it is a convention, taken because it allows to build a scientific corpus that apparentely fits the perceived reality and especially its seemingly continuum aspect. This convention must not be taken as an absolute truth.

After those precisions, let us proceed further. with some other conventions and notations.

In this memoir, all considered fields will be commutative and of characteristic 0 except expressly stated otherwise; all algebraic varieties considered will be defined over such fields. Following ultra standards conventions, we will designate successively by

$\mathbb{Q}$ the field of rational numbers,
$\mathbb{R}$ the field of real numbers,
$\mathbb{C}$ the field of complex numbers and
$\mathbb{Z}$ the ring of integers;

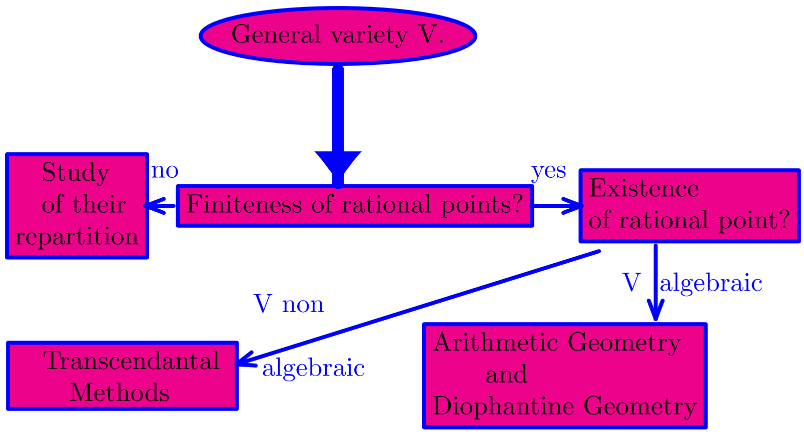
k a field of characteristic 0;
$\bar{k}$ its algebraic closure,
Ω_k the set of its places, that are basically related to its absolute values,
v such a place and
k_v the completion of k relatively to v; that is the metric completion of k equipped with the absolute value defined by v; relatively to Cauchy sequences in k for the metric associated to v.

The ⧫ symbol will mark the end of a proof.

Let k be a field of characteristic 0. If V is a variety defined over k and if K is a field extension of k, we denote by $V(K)$ or V_K the extension of V to K, made up of points of coordinates in K satisfying the initial equations of V with coefficients in k; we will then call $k-$rational points of this extension V_K, the elements of the initial variety $V_K(k) = V(k)$.

In the language of schemes, the algebraic varieties defined over k will be separated and integral (irreducible and reduced) $k-$schemes of finite type.

By the way, we will barely use this language in the on-going versions of this translated memoire since there is no need of using an ultra-heavy digging machinery to only scratch the surface of a well grounded field of research : we will only use but our bare hands with some light tools in this exploratory dissertation, so there will be very few technical proofs but rather more expositions of hopefully interesting original ideas and insights.


General variety V.
Finiteness of rational points?
no
Study
of their
repartition
yes
Existence
of rational point?
V algebraic
Arithmetic Geometry
and
Diophantine Geometry
V non
algebraic
Transcendantal
Methods

Chapter 1

Rational points on algebraic varieties.

1.1 Introduction: State of the Art with Lewis Mordell, Richard Brauer, Yuri Manin and Gerd Faltings.

Lewis J. Mordell, in 1922, proposed a finiteness conjecture that has longly resisted against repeated assaults of the mathematical community : a smooth algebraic curve of genus greater or equal to 2; defined over a number field; admits only a finite number of rational points. Gerd Faltings, using the whole arsenal of modern Arithmetic Geometry mentioned in the introduction, overcame this finiteness conjecture around 1983, about sixty years later, followed about a decade later by Paul Vojta with an Arithmetic-Diophantine Geometry approach combining Diophantine Approximations with Arithmetic Geometry via Arakelov Geometry. Unfortunately, this 20^{th} century great achievement is not an existence theorem : it does not precise if the studied curve admits or not such a rational point.

To decide whether or not, an algebraic variety admits any rational point; we have several tools that provide increasingly finer conditions :

1. The first tool is the Hasse principle, it is a reduction principle based on the adeles space of the field $\mathbb{Q}$ of rational numbers; its implementation, although being a finite process, is not obvious in general. The problem is then also *reduced* to knowing if the studied variety satisfies this "local-global" *reduction* principle of Hasse.

2. The second tool, the Brauer-Manin set, is finer and is built from the Brauer group or Brauer-Grothendieck group of the variety; this finer tool can moreover account for the violation of the Hasse principle, that is; it can also account for the ineffectiveness of the first tool in tackling the studied variety; in the form of an obstruction, called the Brauer-Manin obstruction to the Hasse principle, built from this Brauer-Manin set and often abusively shortened as the Brauer-Manin obstruction.

3. The third toolkit that integrates the first two ones is the even more elaborate notion of Torsors, those allow to build finer filtrations or chain of inclusions starting from the rational points of the studied variety, than the ones built from the preceding tools.

Now, let us present a few contextual words about those criteria.

Soon after the elaboration of the Hasse principle around 1925, the first contradicting varieties were found around 1940 in the form of genus-1 or cubic curves.

Thirty years later, around 1970, the Brauer-Manin obstruction explained almost all those curves counter-examples. But a few decades later, in the late 1990s, the Brauer-Manin obstruction was also found out to be not exclusive; indeed, it appeared that it does not account for all cases of violation of the Hasse principle in the case of surfaces : Alexeı Skorobogatov having published an important article [Sko99] which goes in that direction; proving that even the Brauer-Manin set also fails to account for all the failures of the reduction principle of Helmut Hasse. This article dating from 1999 was firstly submitted in November 1997, as a prepublication [Sko97] of the Max Planck Institute, still available online.

Regarding curves, this exclusivity of the Brauer-Manin obstruction to account for the failure of the Hasse principle, is conjectural; it depends in particular on topological and geometric properties of the curve, like its genus and some dimensional and/or cohomological invariants; that is roughly, its "arithmetic shape", a vague qualitative invariant to be clarified in a hopefully near future.

Finally, for the third tool-kit, even the finest Torsors studied so far also fail to account for all the failures of the Brauer-Manin obstruction. Bjorn Poonen having recently exhibited a counter-example in 2008, following the landmark one of Alexei Skorobogatov in 1998.

To conclude this thread of thoughts, let us notice the increasing complexity of the preceding evoked contractidicting varieties, going along with the increasing complexity of the concerned arithmetic criteria.

1. For the Hasse-principle : some cubic curves (Carl Erik Lind, Hans Reichardt and Ernst Selmer around 1945).

2. For the finer Brauer-Manin obstruction : a bielliptic surface (Alexei Skorobogatov in 1998).

3. For some even finer Torsors obstructions : a 3-fold or solid fibered over a curve into surfaces (Bjorn Poonen in 2008).

Here, an important structuring heuristic pattern emerges from the preceding enumeration : on one side, the "qualitative space" of arithmetic criteria and on the other side, the resulting stratifications of moduli spaces of varieties those criteria generate, or in other words; the moduli spaces stratified by those criteria.

On the concrete side of the corpus; recent computational work has been carried out by Victor Flynn among others; in this direction and then presented at the University of Sydney, during a seminar devoted to this topic in March 2003. The University of Sydney being the alma mater of Magma; the Computer Algebra System (CAS) focusing mainly on Arithmetic Geometry problems.

Since this time and the writing of the first french version of this memoir, the increase of computing performances of machines (higher clocks speeds of both CPUs and buses together with the promising simultaneous synergetic use of both multicores CPUs and GPUs); allowed numerous articles on concrete computations of the Brauer group of relatively tractable varieties to be released; mainly relying on Magma code and routines : some names to be looked for in the Internet; are Martin Bright, Tim Browning, Niels Bruin, Brendan Creutz, Victor Flynn, Ronald Van Luijk, Emmanuel Peyre, Thomas Preu, Samir Siksek, Michael Stoll, Anthony Varilly-Alvarado; and numerous arithmeticians scattered around the world; those may be spotted by just making an Internet search with keywords like "computation of rational points" or so.

Synthetically; the presence of rational points on algebraic varieties is fairly well ruled out for small genus curves, that is up to genus 2; the situation partially darkens for curves of greater genus and almost completely blackouts for varieties of high dimensions. The case of surfaces being an intermediate complexity case between curves and high dimensional varieties.

Note that also within this first and most natural stratification given by the dimension, take place many other inner stratifications, like the genus one in the unidimensional-varieties or curves stratum.

As a partial synthesis, we can point out that we do not have a simple criterion, allowing us to say directly whether or not a random algebraic variety has any rational point; we are just beginning to have a very coarse classification that lists algebraic varieties satisfying the Hasse principle and another listing algebraic varieties that violate this principle, but whose violation are exclusively accounted for by the Brauer-Manin obstruction.

Let us mention, for information — which we will not develop here — that among the numerous means of figuring out the presence of rational points on algebraic varieties, is the method of fibrations often coupled with descent techniques; those approaches are basically "divide and conquer" ones : they consist in tackling rational points on a variety using other ones build from it, for which the problem is expected to be easier; so those techniques consist in splitting a difficult problem into numerous hopefully simpler ones. This is the core of Mathematics problem solving process : finding or building the optimal category shifts giving the most important complexity drops of the initial problem, and eventually combine the results obtained from those shifts to get the initial problem answered.

Finally, we can conclude that both approaches of fibrations and descent methods can be integrated within the already mentionned torsors techniques; that uses them all and seems to be currently the most promising tool; their pinacle being the even more promising motivic ones.

1.2 The Hasse Principle.

In what follows, k will denote a commutative field of characteristic 0, morally; the cases of $\mathbb{Q}$ and that of number fields or finite extensions of $\mathbb{Q}$ will be in mind; that of the field $\mathbb{Q}$ being the most appealing in the daily life of the arithmetician : in the sequel when the base field is not mentioned this will be the tacit cases.

The Hasse principle is a principle belonging to the "local-global" family; it is a fairly general principle in mathematics belonging to the already cited larger class of "divide and conquer" techniques; that allows the understanding of a general complicated situation from its hopefully simpler localized reductions or specializations.

The main contemporary interest of that principle is that it allows to make the problem of the existence of a rational point algorithmically computable : if a variety satisfies the Hasse principle, then there is a theoretically finite execution-time algorithm that tells if the variety has a rational point or not; this computability property giving the most interesting qualitative strata of algebraic varieties in the nowadays digital era.

Let's get back to our initial concern and start with an elementary but important proposition.

Proposition 1. *Let V be an algebraic variety over a field k. If $V(k) \neq \emptyset$ then $V(k_v) \neq \emptyset$ for any place v de k.*

Proof. For any place v of k, we have $k \hookrightarrow k_v$ hence $V(k) \hookrightarrow V(k_v)$. Therefore; if $V(k) \neq \emptyset$ then $V(k_v) \neq \emptyset$ for any place v of k. ♦

If the converse is true, then we say that the variety satisfies the Hasse principle; that is, it admits a rational point globally if and only if it admits a rational point locally for all the places of its base field.

The considered base fields of varieties are the critical objects of this approach; grounding it deeply into the top-layer crust of Field Theory, the latter lying above its even deeper Abstract Algebra underground foundations.

Definition 1 (Hasse principle)**.** *Let V be variety defined over a field k. We say that V satisfies the Hasse principle if it verifies the following implication :*

$$(\forall\, v \in \Omega_k, V(k_v) \neq \emptyset) \Longrightarrow V(k) \neq \emptyset.$$

We can condense this relatively complex formulation through the introduction of the more intricate space of adeles of a variety; transfering to this space the initial complexity of the principle formulation.

This space is built from the ring of adeles $\mathbb{A}_k$ of the base number field k; this ring constitutes a very wide enlargement of k by merging together in a specific manner its metric completions k_v and their local rings O_{k_v}; hence adding a topological oriented oversight and completing the pure algebraic approach with the topological and analytic ones of both $p-$adic Analysis and $p-$adic Geometry.

More precisely $\mathbb{A}_k$ is the subring of $\prod_{v \in \Omega_k} k_v$ constituted of tuples indexed by Ω_k with all but finitely many coordinates in the local rings O_{k_v}, those local rings turn out to be the unit closed balls of the k_v for the absolute value defined by v.

Definition 2. *Let V be a proper and smooth variety, defined over a number field k, we call the space of adeles of V, denoted by $V(\mathbb{A}_k)$ the following product :*

$$\prod_{v \in \Omega_k} V(k_v).$$

Remark. The elements of the space of adeles of a variety V are also called the *adelic points* of V.

Note also that the "proper" hypothesis allows to use "a Tychonov argument" in order to identify the adelic space $V(\mathbb{A}_k)$ genuinely defined from the product over all but finite subsets of places v of k of the corresponding local-rings points $V(O_{k_v})$ of $V(k_v)$; with the preceding product of the $V(k_v)$, denoted $\prod_{v \in \Omega_k} V(k_v)$ over all places v of k : the genuine definition of the adeles space just lifts or spreads out to varieties the construction of the adeles ring $\mathbb{A}_k$ from the base number field k, both constructions being what is called a restricted product.

For specific proper algebraic varieties like classical projective ones given by homogeneous polynomials, the preceding identification of the restricted product with the whole product is relatively easy to establish; for general proper schemes the task is much more tedious.

The space of adeles of a variety V is also called *the adelic space* of V, it is introduced equipped with the induced natural product topology of the whole product, for everything related to *weak approximations* questions : we then say that a variety satisfies the weak approximation if its diagonal inclusion is dense in its adelic space, for this topology.

Thus, a proper and smooth variety defined over a number field, satisfying the Hasse principle admits a rational point if and only if its adelic space is not empty : this is the evocated simplification of this principle formulation.

The reverse of the previous proposition gives an elementary criterion of absence of rational point.

Proposition 2. *Let k be a field of characteristic 0 and V a variety defined over k. If there exists a place v of k such that $V(k_v)$ is empty then V does not admit any k-point.*

The problem is then also *reduced* to finding out whether or not a variety satisfies this basic *reduction* principle of Hasse; it is a still open classification problem, widening the investigations context upto the higher level setting of moduli spaces or spaces of families of varieties.

We have a few families of varieties verifying the cited Hasse principle; the initial result being that of the mathematician who gave his name to it; Helmut Hasse in the footsteps of Adrien-Marie Legendre and Hermann Minkowski, indeed proved the theorem below often suffixed in the litterature as the Hasse-Minkowski theorem, in the beginning of the 1920's.

Theorem 1 (Hasse, 1924)**.** *Let k be a number field. Any quadratic form with coefficients in k vanishes in k if and only if it vanishes in all the completions k_v at all the places v, of k.*

Remark. When a quadratic form defined on a field k vanishes at a point of a extension K of k, we say that the quadratic form vanishes in K or represents zero over K.

Thus, the preceding theorem asserts that any quadratic form defined over a number field k represents 0 over k, if and only if, it represents 0 over k_v , for all places v of k. In terms of varieties, that is our concern; this is equivalent to saying that the family $\mathfrak{Q}_k$ of quadrics defined over a number field k as the $0-$loci of quadratic forms with coefficients in k, satisfies the Hasse principle.

1.3 The Brauer group and the Brauer-Manin obstruction.

Some algebraic varieties contradict the Hasse principle, since that quadrics do not, those contrexamples must be at least cubics : for example the cubic pointed out by Cassels and Guy in 1966 in [CG66] [1]; and given by the equation

$$5x^3 + 9y^3 + 10z^3 + 12t^3 = 0.$$

This failure is often accounted for by one (among others) obstruction, called the Brauer-Manin obstruction, publicly introduced by Yuri Manin in his classic book on cubics first published around 1975 [Man74]; it is an obstruction of cohomological type that involves the Brauer group of the variety.

The main problem is then reduced to computing this group explicitly : generally; it turns out to be a quite tedious task, since whole thesis are dedicated to its computation for relatively simple classes of algebraic varieties, till nowadays. To our knowledge, we have few cases where it is explicitly computed, James Milne [Mil70] did it for some surfaces defined over finite fields, which are outside this memoir framework. Let us also point out an article by Andrew Kresch and Yuri Tschinkel [KT04]

This obstruction dates back to the early seventies. It seems that Yuri Manin was its pioneer; it was then too optimistically thought to account for all cases of violation of the Hasse Principle; unfortunately it was found out later, to be not the case; as we have mentioned it earlier in this memoir introduction. In the case of elliptic curves and more generally that of abelian varieties, the Brauer-Manin obstruction also involves the Tate-Shafarevich group of the variety.

1.3.1 Definitions of the Brauer group.

The classic definition of the Brauer group of a variety, starts from the also classic one of the Brauer group of fields essentially initiated by the german golden age school of Abstract Algebra between the two world wars; this initial stage was developed through the notion of central simple algebras and was generalized to varieties via the spreading-out ability of the notion of Azumaya

[1] This cubic defined over $\mathbb{Q}$ has a non-empty adelic space and no rational point. This absence of rational point escapes any elementary approach of reduction modulo a prime number : the known successful approach follows the paths of Galois theory.

algebras, that emerged and was elaborated after the second world war by Goro Azumaya, one member of the newborn celebrated Abstract Algebra Japanese school of that era.

Before proceeding any further, a few foundational words about classic Brauer groups of fields might be worthy : in Mathematics and perhaps everywhere else, an object of study is often understood through its outside interactions, so what mathematically matters are arrows and their higher versions that are functors. To get core data about an object, nuclear arrows are favored, arrows to nuclear-objects are simpler and more prone to extract the quintessence of the studied object. If a specific structure of the studied object is targeted, arrows that respect it directly or by equivalence, are again chosen within those nuclear ones.

Here for the study of a field k, the algebras-over k are the embodied arrows, since an algebra over k is just the gift of a ring morphism starting from k; the nuclearity of arrows is obtained by restraining to simple and central algebras, that are algebras with both trivial two-sided ideals and center. The structure targeted is the algebro-geometric tensorial one, it is obtained through the constraint of the Morita tensorial equivalence modulo matrices algebras over k; in the end, the Brauer group of the studied field captures the algebro-geometric tensorial complexity generated by that field.

Now for the extended versions of the Brauer group to varieties, the most often found one in research articles —for example, in the landmark article of the trio, Colliot-Thélène, Sansuc and Swinnerton Dyer [[CTSSD87a], p.47] —is the most general; it is the one that applies to any scheme and not only to varieties, it is the definition of the Brauer-Grothendieck group or the Brauer group of étale cohomology, a long term fruit of the mid 1950s golden age french Algebraic Geometry school, lead by Alexander Grothendieck and Jean-Pierre Serre around Paris.

Definition 3 (The Brauer-Grothendieck group)**.** *If V is a $k-$variety, the Brauer-Grothendieck group of V noted $Br(V)$, is the following second group of étale cohomology :*

$$Br(V) = H^2_{ét}(V, G_m)$$

where G_m is the sheaf of multiplicative units of the structural sheaf of V, represented by the group scheme $Spec\, \mathbb{Z}\left[T, \frac{1}{T}\right]$.

However, in the article [[EHKV01]], and the corresponding prepublication [[AG03-01249], p.10], it is specified that we just have an injective morphism, called the Brauer morphism; from the classical Brauer group defined by Azumaya algebras into the torsion of the previous étale cohomology group. This morphism is surjective for some classes of schemes (spectra of peculiar rings, schemes satisfying specific connectedness and regularity conditions and some other cases).

In this same article, they define the cohomological Brauer group as the torsion previously evoked; contrary to the conventions of James Milne [[Mil80], p.147] and those of Alexander

Grothendieck [[Gro68], p.80], who call the cohomological Brauer group the whole étale group in question : the clue is that surely this one should be a torsion group in the general case; so equal or at least isomorphic to its torsion subgroup.

In a narrower setting the Brauer group of a variety can be defined using the theory of local fields ("à la Serre" with its « ramification » aspect) as well as the cohomology of the corresponding groups applied to the localizations of the function field of the variety. This is the point of view of Serge Lang [[Lan91],p.251].

Let us point out another way of seeing things. Yuri Manin, one of the PHD students of Igor Shafarevich, defines the Brauer group of a variety in Galois terms, by considering the extension of the variety over a Galois extension of its base field, and the cohomologies of the Galois group of this extension with coefficients in the groups of divisors of the extended variety.

The work would be to see how all these definitions fit together; and to inspect closely the seemingly different étale definitions. Note that we did not have access to the important resource that constitutes the recent book by Alexei Skorobogatov [Sko01].

A decade and a half after the first french version of this memoir, we could have a look at it; this monograph actually appears to be now, the reference book on torsors; those are the fruitful "group-schemes tool" that combine groups actions and schemes to build obstructions sets from the studied (class of) varieties : stacks and gerbes encompass the next-level similar combination process.

The author of this book and another expert on the subject, namely Jean-Louis Colliot-Thélène; have recently written in 2019 what might become in a near future another reference book [CTS21] on Brauer groups clarifying all the queries about them raised here; again the curious reader may find digital preliminary versions of that book, by searching the Internet for the first author name.

It is Manin point of view that we will adopt here. Why? Because of the Galois perspective, having in mind the motivic Galois groups in a hopefully near future. We place ourselves within the framework of the introduction notations, V denotes a variety defined on a field k of characteristic 0 with a Galois extension $K \hookleftarrow k$ of Galois group G.

We have an action of G on the groups $K(V)^{\times}$ and $Div\,(V_K)$, respectively the multiplicative group of non-zero elements of the function field $K(V)$ of V_K and the group of Cartier divisors of V_K; this action induces a morphism of cohomology groups of G, as follows.

$$H^2\left(G, K(V)^{\times}\right) \xrightarrow{\varphi_{V_K}} H^2\left(G, Div\,(V_K)\right).$$

With the previous notations, we have the following equality.

$$Br(V, K) = \ker \varphi_{V_K}.$$

Remark. This is the exact definition given by Yuri Manin in his reference book on cubic surfaces. [[Man74], p.221]. Note that this $Br(V, K)$ is intrinsically not $Br(V)$, the Brauer group of V since it is built from a base field change from k to a peculiar Galois extension K. To give a rough idea of how one gets $Br(V)$, the Brauer group of V, the reader may think of it as the merging along Galois extensions K of the base field k of those relative Brauer groups $Br(V, K)$: the advantage of the Grothendieck approach is that its étale definition encapsulates the Galois process involved here; in the same manner the introduction of the intricate adelic space encapsulates the complexity of the Hasse principle formulation.

We are going to explicit this definition a bit. The group of invertible elements of $K(V)$ is mapped injectively by a canonical morphism θ_V into $Div\,(V_K)$, the group of Cartier divisors of V_K, giving the following inclusion.

$$K(V)^\times \overset{\theta_V}{\hookrightarrow} Div\,(V_K).$$

As θ_V is a $G-$equivariant morphism, the next functoriality proposition of group cohomology ensures that θ_V induces a morphism $H^n(\theta_V)$ between all the corresponding cohomology groups of G, giving the following arrow.

$$H^n(G, K(V)^\times) \xrightarrow{H^n(\theta_V)} H^n(G, Div(V_K)).$$

The morphism φ_{V_K} is obtained for $n = 2$.

Proposition 3. *Let G be an abelian group; A and B two G-modules; then any injective $G-$equivariant morphism ϕ from A to B*

$$A \overset{\phi}{\hookrightarrow} B$$

induces for each index value n, a morphism $H^n\,(\phi)$ between the corresponding cohomology groups of G as in the following arrow.

$$H^n\,(G,\,A) \xrightarrow{H^n\,(\phi)} H^n\,(G,\,B).$$

Proof. The strategy of the proof splits into three stages. First, we write down the complexes

$$\mathbf{C}_A = \left(C^n(A), d^n_A\right)_{n\geq 1}$$

and

$$\mathbf{C}_B = \big(C^n(B), d_B^n\big)_{n\geq 1}$$

associated to the two G-modules A and B; then secondly, we notice that the compatibility of the action of G with ϕ implies the existence of a morphism $\phi_\bullet$ between these two complexes, out of which, thirdly and finally; we derive the desired morphisms between the cohomology groups. Recall that $C^n(A)$ is the set of maps from G^n to A and that the coboundary map $d_A^n \,:\, C^n(A) \longrightarrow C^{n+1}(A)$ is defined by the following expression.

$$d_A^n(f)\big(\mathrm{g}_{n+1}\big) = g_1 f(g_2\,,\ldots,g_{n+1}) + \sum_{i=1}^{i=n}(-1)^i f(g_1\,,\ldots,g_i g_{i+1}\,,\ldots,g_{n+1}) + (-1)^n f(\mathrm{g}_n)$$

with $\mathrm{g}_n = (g_1\,,\ldots,g_n)$.

If f is in $C^n(A)$ then its composition with ϕ noted $\phi_\bullet(f) = \phi\circ f$ allows to define a map $\phi_\bullet$ from $C^n(A)$ to $C^n(B)$. The compatibility of the action of G with ϕ gives the equality $\phi_\bullet\circ d_A^n = d_B^n\circ\phi_\bullet$. This means exactly that we can define a morphism of complexes, illustrated by the following diagram.

$$\begin{array}{ccccccc}
\mathbf{C}_A\ldots & \longrightarrow & C^n(A) & \xrightarrow{d_A^n} & C^{n+1}(A) & \longrightarrow & \ldots \\
\big\downarrow \phi_\bullet & & \phi_\bullet\big\downarrow & & \phi_\bullet\big\downarrow & & \\
\mathbf{C}_B\ldots & \longrightarrow & C^n(B) & \xrightarrow{d_B^n} & C^{n+1}(B) & \longrightarrow & \ldots
\end{array}$$

As ϕ is injective, $\phi_\bullet$ is also injective and we can then consider that the coboundaries (respectively cocycles) of the first complex form a subspace of the coboundaries (respectively cocycles) of the second one. We can then define the desired morphisms $H^n\,(\phi)$ between the corresponding cohomology groups. ♦

Remark. Those general very basic tools of Group Cohomology outlined in the preceding proposition should be kept secured inside one drawer of the arithmetician toolbox; they were used to build the first stepping basement of modern Arithmetic Geometry; resulting in the first "Linearization floor"; one of the top Linearization floors being the panoramic Motivic one. Paradigmatically, reversing the metaphore the other way-around may be more correct : the first linearization underground basement of Arithmetic Geometry being the Homology Algebra one and the deepest rooted one being the Motivic basement.

As an application-example of those Cohomology tools use in the Brauer theory sketched here; we can mention the first Galois cohomology group of a suitable complex with coefficients in the Picard group of the extended variety $V_{\bar{k}}$: this Galois cohomology group is related to one Brauer invariant of V; namely the algebraic part of the Brauer group of V, often denoted $Br_1(V)$.

Once $Br(V)$ the Brauer group of a variety V is defined; let us pinpoint from the now abundant litterature about them, its main relevant properties for computational purposes about rational points study on sufficientely amenable varieties V, like smooth projective ones :

1. $Br(V)$ is a birational invariant.

2. $Br(V)$ is a torsion abelian group.

3. $Br(V)$ is a subgroup of the Brauer group of the field of functions of V; in short terms we have the inclusion
$$Br(V) \subset Br(k(V)).$$

The first property, allows to drop the complexity of its computation by scanning the space or moduli of varieties birationally equivalent to the studied one, looking for the ones giving the easiest computation in the spirit of the Minimal Model Program of the large Birational Geometry corpus of Algebraic Geometry : this very classical birational process is already often used to lower the degree of the concerned varieties when directly looking for rational points; for instance, in the case of algebraic plane curves, it generally allows to switch from a quartic downto a cubic or even a quadric.

The second property provides a mean for shifting the study of the Brauer group to the better known category of torsion abelian groups, and take advantage of the latter primary decomposition techniques : in most recent thesis on Brauer group computations, the determination of the $p-$torsion part of the Brauer group for small prime p is often combined to (mostly elliptic) fibration methods to get some clues about the Brauer group of the studied variety from the $p-$torsion part of the Brauer groups of the (elliptic) fibers.

Finally the third property allows to locate and bound the Brauer group of a variety inside the more accessible one of a field namely its functions field, in a kind of minimum-locating category-shrink process applied to the argument of the Brauer group functor.

1.3.2 The Brauer-Manin obstruction.

Let us recall that the main interest of the Brauer group of a variety V is that it can indicate the absence of rational points on V and also be used, as Yuri Manin did in the 1970s, to build a filtration able to often account for the violation of the Hasse principle by V when looking for those rational points.

Manin's idea starts from the natural diagonal inclusion of $V(k)$ into its adelic space :

$$V(k) \hookrightarrow V(\mathbb{A}_k).$$

Then, construct an intermediate variety, from the Brauer group of V, which we denote by $V(\mathbb{A}_k)^{Br(V)}$; refining the preceding chain of inclusions, giving the finer inclusion chain below :

$$V(k) \hookrightarrow V(\mathbb{A}_k)^{Br(V)} \hookrightarrow V(\mathbb{A}_k).$$

If $V(k)$ is empty without $V(\mathbb{A}_k)$ being so, V violates the Hasse principle; in this case, if we already had $V(\mathbb{A}_k)^{Br(V)}$ empty, we say that the violation of the Hasse principle is accounted for by the Brauer-Manin obstruction.

If $V(\mathbb{A}_k)^{Br(V)}$ is empty then V does not admit any rational point, we then say that there is a Brauer-Manin obstruction to the existence of a rational point; so that the Brauer-Manin set can pinpoint two arithmetic failures of the variety V : its failure of having a rational point and also its failure of satisfying the Hasse principle when looking for those rational points.

The intermediate variety in the above chain, allowing these results is constructed from a *coupling* resulting from — among many facts— a property of functoriality of the Brauer-Grothendieck group; indeed, for any field K over which V is defined, we have the following fundamental coupling, often called the Brauer-Manin pairing, that generalizes the Cassels-Tate pairing involving the Tate-Shafarevich group in the case of elliptic curves.

$$V(K) \times Br(V) \xrightarrow{e_K} Br(K).$$

To give some general ideas of where this coupling comes from, one should think of it as a specialization or evaluation map consisting in evaluating a class of the Brauer group of V_K defined roughly from $Br(K(V))$, the Brauer group of the functions field of V_K; at points of V_K by taking the values of those functions at those points, landing down to a class in the Brauer group of the base field K.

We now assume that k is a number field. The Class Field Theory then gives two results concerning the Brauer group of k. The first one is a morphism, for each place v of k, called the local-invariant of Hasse associated to v and written as follows.

$$inv_v : Br(k_v) \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

To give again some rough idea of where this map comes from, let us recall the obvious following two parallel matching short exact sequences, the second one being of a sheaf-functor nature.

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

$$1 \longrightarrow \mathbb{G}_m \longrightarrow GL \longrightarrow PGL \longrightarrow 1.$$

One gets the Hasse local-invariant by taking the Galois cohomology groups of $\bar{k_v}^s$, the separable closure of k_v, with coefficients in the sections of the nodes of the second sequence at $\bar{k_v}^s$ and ending-up with a long derived exact sequence that simplifies through Class Field Theory theorems; then by mapping the results into the nodes of the first exact sequence, that finally give the wanted local-invariant of Hasse through Class Field Theory main theorems again.

This technique is an instance of the effectivity of the cohomology toolkit mentioned earlier after the Brauer group definition, one might even use the terms of powerful linearization machinery, since it is able to produce quickly results via simplification of long or short sequences coming from the translation of classical theorems into the language of cohomology.

The second result that Class Field Theory gives is another less obvious exact sequence, consequence of the law of reciprocity, noted as follows :

$$0 \longrightarrow Br(k) \longrightarrow \bigoplus_{v \,\in\, \Omega_k} Br(k_v) \xrightarrow{\sum \mathrm{inv}_v} \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

We get from these two results of Class Field Theory the following critical coupling :

$$V(\mathbb{A}_k) \times Br(V) \xrightarrow{e} \mathbb{Q}/\mathbb{Z}.$$

In more precise terms, for $\mathbf{a} = (\mathbf{a}_v)_{v \,\in\, \Omega_k} \in V(\mathbb{A}_k) = \prod_{v \,\in\, \Omega_k} V(k_v)$ and $\alpha \in Br(V)$ we have the following expression :

$$e(\mathbf{a}, \alpha) = \sum_{v \,\in\, \Omega_k} \mathrm{inv}_v \circ e_{k_v}(\mathbf{a}_v, \alpha).$$

Note that the properness of the variety must be again invoked to ensure the legitimity of the previous sum over all places of k of the local-invariants values taken at the corresponding evaluations maps $e_{k_v} : V(k_v) \times Br(V) \longrightarrow Br(k_v)$ outputs.

We then define $V(\mathbb{A}_k)^{Br(V)}$ as being the « orthogonal» of $Br(V)$ relatively to this coupling or as the left kernel of this coupling; in the following formulation.

$$V(\mathbb{A}_k)^{Br(V)} = \Big\{ \mathbf{a} \in V(\mathbb{A}_k) \mid \forall\, \alpha \,\in Br(V),\ e(\mathbf{a},\ \alpha) = 0 \Big\}.$$

This is briefly what can be said about the Brauer-Manin obstruction, the main challenge being to compute explicitly the algebraic objects coming into play, in particular for classes of particular varieties. This is exactly what Martin Bright did in his thesis and two of his articles about a family of quartics, [Bri05] [BSD04].

Another text on computations is the recent thesis of Thomas Preu defended at the ETH of Zurich; where the Brauer groups of some other quartics are computed, mainly using Magma routines; the curious reader might look for it in the Internet by searching its author name. As already mentionned, this task is not straightforward since whole PHD-thesis are dedicated to it, till now.

Remark. $V(\mathbb{A}_k)^{Br(V)}$ is sometimes referred to as the Brauer-Manin set of the variety V. More elaborate methods, from the already cited theory of torsors, make it possible to construct more detailed (finer and longer) filtrations or chains of inclusions than the one resulting from solely the Brauer-Grothendieck group.

The theory evoked here is strongly based on the fact that the studied varieties are algebraic; as its objects, its methods are also strongly rooted in Abstract Algebra and a considerable amount of its ramifications. Those deep Abstract Algebra foundations have one key advantage : the rigidity of Pure Algebra methods is intrinsically inclined towards algorithms computations on machines.

Nonetheless, there is a principle in Number Theory claiming that the arithmetic of a variety is governed by its geometry, in particular its differential geometry; one can then legitimately ask how to extend the study of rational points to differentiable varieties, a class larger than that of algebraic varieties.

This is what we will consider in the second part, by stating then testing a conjecture, which — to our knowledge — is not in the literature; this conjecture establishes a seemingly new connection between two intricate fields of investigations of Number Theory, namely the question of the existence of rational points on varieties and the theory of algebraic independence of numbers.

A common point between the two parts may be found in the fact that for some real algebraic varieties, meaning varieties defined over the field $\mathbb{R}$ of real numbers, the Brauer group reflects the standard topology of the variety that is, its topology as a subspace of the standard euclidean space $\mathbb{R}^n$ for some integer n; indeed for real curves — see the article by Frank De Meyer and Max-Albert Knus [DK76]— and for some real surfaces — see the article by Viacheslav Nikulin [Nik94]—; the Brauer group is strongly related to the number of connected components of the variety; just like the first De Rham cohomology group of a real manifold.

Chapter 2

Rational points on general varieties.

Here begins the totally original part of this memoir : its aim is to propose an extension of the study of rational points to general non-algebraic or transcendental varieties. In order to do so, two options appear at first sight :

1. either extend the algebraic Brauer theory of the first part to transcendental varieties or
2. start from the already available Transcendental Numbers Theory.

From now, we will try to trace an unexplored path from the second option by using the Theory of Periods as well as its Transcendental corpus as digging tools, those having recently been sharpened by the modern Arithmetic Geometry and Motivic Geometry grinders. Let us notice that the first option is actually under development, some recent works have been done in this direction through the paths of Gerbes and Stacks.

The advantage of the second option is that Transcendental theories have a quite long history of more than an half-dozen centuries, dating back at least to Gottfried-Willhelm Leibniz and Isaac Newton; so that their maturity might make them more powerful.

We will fix the notations and the situation, and now place ourselves within the framework of real differential geometry for the studied varieties; extending the initial algebraic range of investigations. So, here the studied variety V will be a real differentiable hypersurface of the standard finite dimensional real euclidean spaces $\mathbb{R}^n$, not necessarily algebraic but *enclosed*, in particular bounded. We mean the qualifier *enclosed* to be closed in the geometric sense, as follows.

Definition 4 (Enclosed variety)**.** *A subspace or subset of the real ambient euclidean space will be said* enclosed *if it is ≪ folded back into itself ≫.*

Remark. For some sufficiently smooth varieties, a positive mean curvature might be enough to achieve this requirement.

EXAMPLES. In the case of hypersurfaces or equivalently, codimension-1 subvarieties of $\mathbb{R}^n$; a simple loop in the real $2-$dimensional euclidean plane for the case $n = 2$ or a soap bubble in the real $3-$dimensional Euclidean space for the case $n = 3$.

The studied variety will also be assumed closed for the usual euclidean topology; and it will be assumed ≪ not punctured ≫, that is to say ≪ without holes ≫, so that it constitutes a ≪ continuum ≫.

Definition 5 (Clustered variety)**.** *A subspace or subset of the real ambient euclidean space will be said clustered or ≪ strongly connected ≫if it has no boundary and if each of its proper part admits at least one non-trivial or proper simply connected neighborhood.*

Remark. In other words, a variety is clustered if it is strongly locally simply connected but not globally simply connected : this condition might be fulfilled by requiring that the first fundamental group of the variety is locally trivial.

If V is any variety defined over the field of complex numbers $\mathbb{C}$ and k is any subfield of $\mathbb{C}$, then we will call $k-$points of V and denote them by $V(k)$ the subset of V consisting in its points having all their coordinates in k. We will also denote by $\dot{V}(k)$ the non-trivial $k-$points of V, consisting of points of V with non-zero coordinates in k.

Rational points will implicitly mean $\mathbb{Q}$-points, thus obtained for $k = \mathbb{Q}$; similarly the qualifier ≪ algebraic ≫ alone will refer to ≪ algebraic over $\mathbb{Q}$ ≫ and the same holds for the qualifier ≪ transcendental ≫, so that when the field of reference is not explicitly mentioned, it will implicitely be $\mathbb{Q}$.

Definition 6 (Center of a variety)**.** *The mean point of a clustered and enclosed hypersurface V is the center of gravity or barycenter of the variety V denoted G_V, the mass density being assumed constant in V so that the mass is evenly distributed within V.*

EXAMPLE. If V is a differentiable and enclosed plane curve or a simple loop, $\lambda(V)$ denoting its length, the mean or average point of V is given by the following integral formula, where M is the point running along V during the integration process :

$$G_V = \frac{1}{\lambda(V)} \oint_V \mathbf{M}\, d\lambda. \tag{2.1}$$

Remark. The previous formula giving the mean point extends to any higher dimensional hypersurface V that is clustered and enclosed. Some authors use the term *centroid* instead of *center* for the mean-point.

Now we will introduce the key objects of our original approach in the form of periods defined from the variety. Before that let us precise some terminology.

We will say that a variety W is the bordering ("variété bordante" in french) of another one, noted V, if the topological border of W is exactly V, or if $V = \partial W$, which we also denote by $W = \int V$.

Here, in the spirit of the Jordan theorem about simple plane loops; if a variety V of the standard euclidean space $\mathbb{R}^n$ is *enclosed*, it has two bordering varieties : a bounded inner or inside one $\oint V$ and an outer or outside one $\int V$ that may turn out to be unbounded. In this memoir, the considered bordering or primitive variety will be $\oint V$, the inner bounded one.

Definition 7. *If V is a real, clustered, enclosed hypersurface and if $\oint V$ is its inner bordering bounded variety, we call fundamental arithmetic periods of V the pair of periods*

$$\mathcal{P}_{\mathcal{A}}(V) = \big(\mu_V, \lambda_V\big)$$

with $\mu_V = \iint_{\oint V} d\mu$ and $\lambda_V = \oint_V d\lambda$, where $d\lambda$ is the volume form of V and $d\mu$ that of $\oint V$.

Remark.

Completing this definition, with regard to the working hypothesis, the first *clustered* hypothesis legitimizes the computations of the mean point and of the second period, that is the intrinsic measure of the variety : punctures and even more likely holes on the variety may leave the first period unaffected while allowing an uncountable possible values for this second period; thus compromising a firm definition of the couple or pair formed by those two quantities.

The second *enclosed* hypothesis allows us to compute the measure of the space circumscribed by the variety, that is the first period and ensures the finiteness of the results of the computation of both periods.

Examples.

1. If V is the circle centered at the origin of radius r as in the picture (2.3), $\oint V$ is the closed disc of the same radius centered at the origin , G_V is the origin and $\mathcal{P}_{\mathcal{A}}(V) = (\pi\, r^2,\ 2\pi\, r)$.
2. If V is the hollow torus or "tire tube" of radii R and r centered at the origin and pictured there (2.1) , $\oint V$ is the solid torus or "filled donut" centered at the origin and of the same

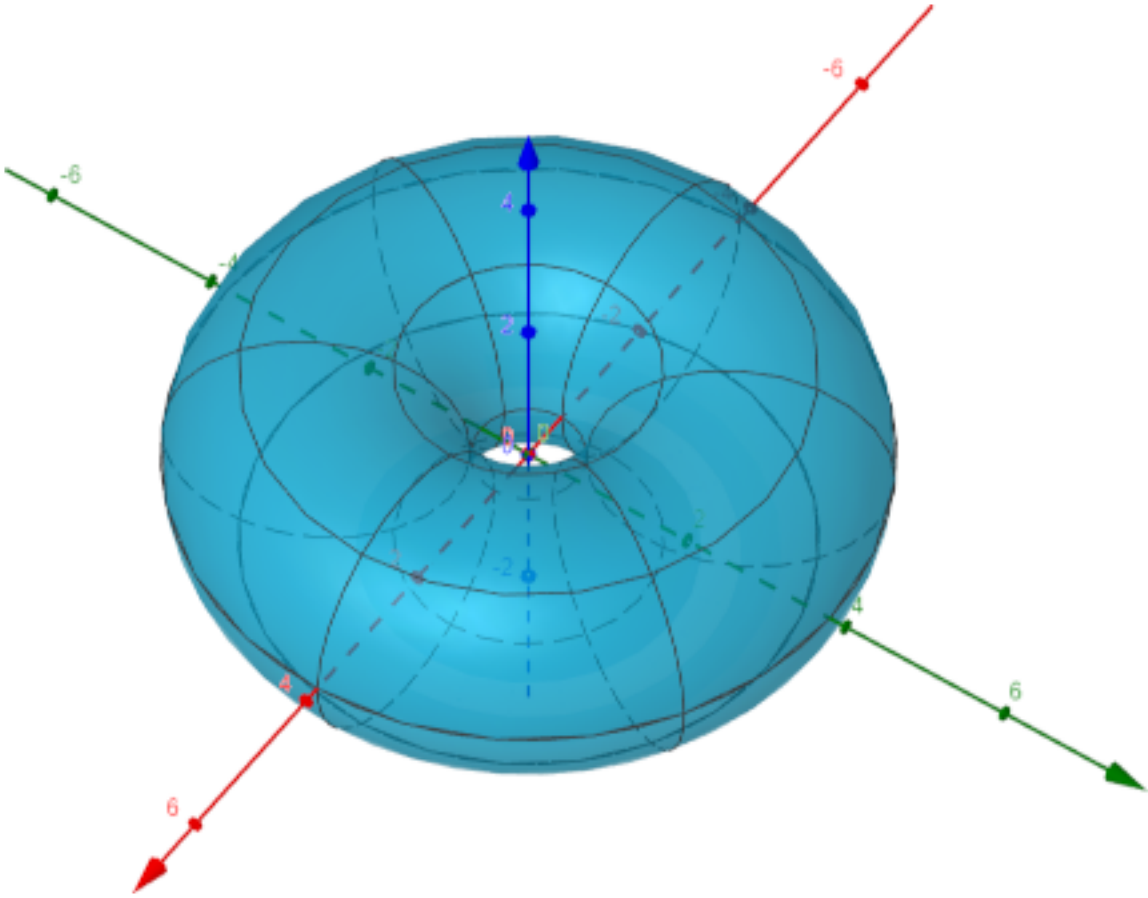


Figure 2.1: A torus of radii R and r.

radii; G_V is the origin and $\mathcal{P}_{\mathcal{A}}(V) = (2\pi^2 Rr^2,\ 4\pi^2 rR)$. This torus is an instance of a locally simply connected variety that is not globally simply connected.

2.1 Statement and study of the criteria conjectures.

In this section, we enumerate a series of conjectures, in the form of criteria telling whether or not a variety admits a non-trivial rational point. These conjectures are increasingly strong, following the footsteps of the author during his investigations. They link the possible presence of non-trivial rational points to questions of algebraic independence of periods; to the best of our knowledge, this has not yet been considered so far.

2.1.1 Conjectural criteria for rational points on general varieties.

The first conjectural criterion is the weakest, we give it as an indication of the process followed during past investigations; it was the first studied object. In the form given here; it will be left aside soon after, since this first version of this weak criterion quickly collapses against simple cases as we will see later. Before that, we call a variety V *principal* if it is given by only *one* equation that may it be algebraic or not.

Conjecture 1 (Weak criterion)**.** *Let V be a real clustered, enclosed and principal hypersurface. Suppose that its mean point G_V is algebraic over $\mathbb{Q}$. If, on the contrary, $\delta_V = \frac{\mu_V}{\lambda_V}$ is transcendental then V has no non-trivial $\mathbb{Q}-$point.*

Remark. In other words, this weak criterion claims that if V admits a non-trivial rational point, then δ_V is algebraic. Note also that the algebraicity of δ_V is equivalent to an *homogeneous* algebraic dependence of both periods components of $\mathcal{P}_{\mathcal{A}}(V)$, that are μ_V and λ_V.

As indicated previously, this version of the weak criterion is quickly put in default in front of elementary cases, as we will soon see. The idea to strengthen it, is to not require the algebraic relation linking the two periods to be *homogeneous*. We then get a condition of general algebraic independence of periods. Here is a little more sharpened statement.

Conjecture 2 (Stable criterion)**.** *Let V be a real, clustered, enclosed and principal hypersurface; with an algebraic mean point G_V. Suppose also that its fundamental arithmetic periods form an algebraically independent family over $\mathbb{Q}$ denoted by $\mathcal{P}_{\mathcal{A}}(V)$, then V has no non-trivial $\mathbb{Q}-$point.*

Finally, the following last criterion is the most general since we do not require anymore the global hypothesis of being principal for the variety V. It will also be quickly reconsidered in the preliminary version given here, because; in that form, it falls apart against some simple counterexamples; extended versions of all the criteria of this memoir are being studied and developed; from the recompilation of piles of past research paper notes that have been accumulated along half a decade between 2004 and 2009. Up to now; the extended versions being elaborated seem quite robust so far; giving a hopefully definitive answer to the question of the presence of rational points on some peculiar classes of general varieties.

Conjecture 3 (Generalized criterion). *Let $(V_1, \ldots, V_n)$ be a finite number of disjoint pieces of principal hypersurfaces. Suppose that their union $V = \bigcup_{i=1}^{i=n} V_i$ constitutes a clustered and enclosed hypersurface, with an algebraic mean point G_V and that the fundamental arithmetic periods of V denoted by $\mathcal{P}_{\mathcal{A}}(V)$ form an algebraically independent family over $\mathbb{Q}$ then V has no non-trivial $\mathbb{Q}-$point.*

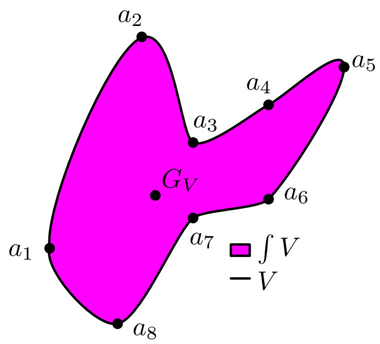


Figure 2.2: Clustered and enclosed hypersurface.

The picture (2.2) represents a variety V being a plane curve, obtained by the continuous gluing of eight differentiable arcs. It turns out that V is enclosed and clustered; we can, therefore, compute its two fundamental arithmetic periods and its mean point. At the time when these lines were written, only the stable criterion in the version given here, resisted a bit against counter examples.

As we have already mentioned, this is the converse implications of the criteria that will be of interest to specialists in the Theory of Transcendental Numbers : if we know that the variety admits non-trivial rational points then we deduce the algebraic dependence of its associated fundamental arithmetic periods.

Remarks.

1. The stated criteria just gives a sufficient condition for the absence of non-trivial rational points : it turns out indeed, that we have elementary counter-examples of the same kind as the weak criterion counter-examples, to the necessity of this condition.

2. The last two ones can be also reformulated in terms of transcendental degree :

$$trdeg_{\mathbb{Q}}\mathbb{Q}(\mathcal{P}_{\mathcal{A}}(V)) \quad maximal \qquad \Rightarrow \qquad \dot{V}(\mathbb{Q}) = \emptyset.$$

In other words, if the transcendental degree of the field generated by the fundamental arithmetic periods of V is maximal, then V does not have any non-trivial rational point. If we allow quotients of periods or generalized periods, then by taking $\mathcal{P}_{\mathcal{A}}(V) = (\delta_V)$, the weak criterion can also be formulated in those terms of transcendental degree.

3. The fundamental arithmetic periods are intrinsic ones, meaning that they are built from integrals along V; indeed $\mu(V)$ initially built from integration along $\oint V$ may actually be obtained by Stokes Formula [1] from integration along V, bearing in mind that $\partial \oint V = V$.

4. The stable criterion is a special case of the generalized one, obtained with $n = 1$ piece.

2.1.2 Synthesis.

As a conclusion, we can say that the criteria main ingredient is a subset of generalized periods of the studied variety and that those criteria claim that the nature of the algebraic relations governing some components of this subset; translated in transcendental degree terms, determines the arithmetic of the studied variety; in particular the presence of non-trivial rational points.

This would provide a quite deep linearization of the problem of studying rational points on varieties; dropping drastically its complexity, because as it might not seem at first sight, algebraic dependence issues are in fact linear dependence ones : an algebraic relation between a tuple or finite set of numbers is just a linear relation of the monomials formed from those numbers. In contemporary terms of computability this is a significant gain.

On the periods side, since the transcendence degree of a tuple of complex numbers or precisely the transcendental degree of the field generated by such a tuple is invariant under birational transformations and more generally under algebraic transformations involving radicals; working-out those criteria will mainly consist in birationally and algebraically simplifying the tuple of fundamental arithmetic periods to get close to a definitive conclusion, as surely plausible; those simplifications of the tuple of fundamental periods should have geometric or more likely motivic interpretations for the studied variety and its associated motive.

On the variety side, bearing in mind that having rational points for varieties is obviously a birational property, another critical issue will be ensuring that birationally equivalent varieties

[1] One of the most important formula of all (Motivic) Mathematics.

give the same transcendental degree for the fields generated by the corresponding arithmetic fundamental periods.

Those two ontological issues of invariance seem to be actually the most critical theoretical parts for establishing the rather unusual set of criteria proposed here.

2.2 Application to some elementary cases.

Similarly to the case of algebraic varieties with the determination of objects relating to the Brauer-Manin obstruction; the explicit computation on concrete cases of objects relating to those criteria is not an easy task in general.

It is essentially a matter of determining the periods involved, together with the mean point; and all of those computations amount to integral and differential calculus on varieties; while having observed previously that all varieties considered hereafter are enclosed, clustered and principal. The advantages of this approach over the Brauer theory computations are multiple :

1. an extremely rich theory of integro-differential calculus developed over past centuries is available to compute those periods, either numerically or explicitly through computer algebra systems.
2. it is a more straightforward approach, simpler and accessible to a wider audience of mathematicians than the Brauer theory one; moreover mathematically speaking and as already mentioned,
3. it proposes to uncommonly bridge two separated branches of Number Theory : on one side, the theory of periods together with the theory of their algebraic independence and on the other side, the theory of rational points on varieties; and finally
4. it allows to extend those rational points investigations to a much larger class of varieties than the classical algebraic ones.

Having those seducing features in mind; and in order to stick to reality and not be naively dreaming; we will benchmark those criteria by testing them on some concrete very elementary cases and see what can be extracted from those experiments.

2.2.1 Elementary algebraic cases : some plane conics.

First things first, we will test the criteria against plane conics, following mathematics chronological order. Those plane conics seem to be the first algebraic varieties ever studied by mankind through purely geometric means; namely in terms of plane sections of cones; or through greek Classical Geometry constructions with ruler and compass.

They are 1−dimensional varieties of degree 2 embedded in a plane; or equivalently, plane curves given as the 0-locus of degree 2 polynomials in 2 variables. Some call them plane quadrics, as zeros locus in the plane of quadratic polynomials.

They are now often qualified as trivial curves by modern arithmetic geometry experts, since they belong to the genus-0 category of curves for which everything is under-control; because for such genus-0 curves either they have no rational point or they have at least one rational point; in the latter case, they are birationally equivalent to the rational line, so they have an infinity of rational points.

Circles. Counterexample to necessity condition of the criteria.

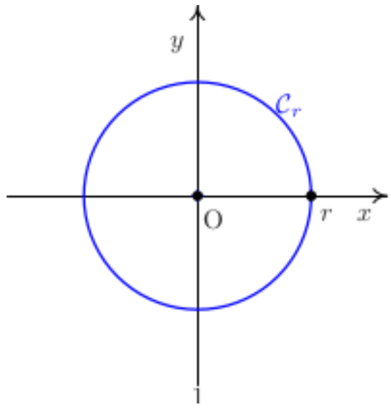


Figure 2.3: Circles $(\mathcal{C}_r) : x^2 + y^2 = r^2$.

We take the previous elementary example of the family of circles $(\mathcal{C}_r)$ centered at the origin and of radius r, we saw that $\lambda(\mathcal{C}_r) = \lambda_r = 2\pi\, r$ and $\mu(\mathcal{C}_r) = \mu_r = \pi\, r^2$, so that we obtain

$$\mathcal{P}_{\mathcal{A}}(\mathcal{C}_r) = (\mu_r, \lambda_r) = (\pi\, r^2, 2\pi\, r).$$

The mean points of those circles $\mathcal{C}_r$ being the origin, they are indeed algebraic. We fall under the hypotheses of the criteria. The algebraic independence of the two arithmetic fundamental periods is equivalent to that of π and r, in other words, we have the following transcendental equality

$$trdeg_{\mathbb{Q}}\mathbb{Q}\left(\mathcal{P}_{\mathcal{A}}(\mathcal{C}_r)\right) = trdeg_{\mathbb{Q}}\mathbb{Q}\left(\pi, r\right)$$

We can then derive the next two cases from that transcendental equality.

1. If we take r transcendental and algebraically dependent of π; for example, a rational power of π like $r = \pi^{\alpha}$ for some $\alpha \in \mathbb{Q}$ with $\alpha \neq 0$; we have the algebraic dependence of the periods without any rational point on the variety since r is transcendental; so we can not therefore have an equivalence or an "if and only if" statement in the criteria in the form given here; this provides a simple counterexample to the necessity previously mentioned, in the form given here, we can hope to have only a sufficient condition of absence of non-trivial rational point.

2. On the other hand; if we choose r algebraically independent of π, we get the algebraic independence of the fundamental arithmetic periods; and the transcendence of r ensures the absence of rational points, confirming thus the stable criterion.

Now a few remarks about this first experimental test of the criteria, before continuing any further, those remarks being useful for the next treated cases.

The first one is to notice that, since the set of complex numbers that form with a fixed transcendental number an algebraically dependent pair of numbers, is countable; the first cases constitute the exceptional or degenerate situation; whereas the second ones that give an algebraically independent pair of numbers; form an uncountable set and constitute the rule or the general situation; so that the initially sought equivalence condition versions of the criteria are only contradicted by the rare and exceptional degenerate situations.

The second one is to note that the version given here of the weak criterion applies well in this case; since $\delta(\mathcal{C}_r) = \delta_r = \frac{\mu_r}{\lambda_r} = \frac{1}{2}r$. So that in terms of transcendental degree we obtain

$$trdeg_{\mathbb{Q}}\mathbb{Q}(\delta_r) = trdeg_{\mathbb{Q}}\mathbb{Q}(r).$$

If $\delta(\mathcal{C}_r) = \delta_r$ is transcendental then r is transcendental and therefore, $\mathcal{C}_r$ has no rational point.

To finish this section in a more constructive manner, let us give back to the reader, for the time taken to reach those lines, one proof or at least a sketch of a proof about the countability fact asserted above :

Proposition 4. *If τ is any transcendental complex number, the set of complex numbers z that form with τ an algebraically dependent pair of numbers (τ, z) is countable.*

Proof. The strategy proposed to get that elementary result consists in three steps.

1. First notice that τ being transcendental, the pair (τ, z) is algebraically dependent over $\mathbb{Q}$ if and only if the number z is algebraic over $\mathbb{Q}(\tau)$.

2. Remark that if k is a countable subfield of $\mathbb{C}$ then the set of complex numbers that are algebraic over k is countable. This is because such a set is a countable union of countable subsets of $\mathbb{C}$; indeed it is the union over the degree d of the sets of algebraic numbers over k of degree d.

3. Conclude by noticing that $\mathbb{Q}(\tau)$ is a countable subfield of $\mathbb{C}$.

The reader should be able to gather those arguments to merge a proof of the stated fact, and be good to go through next sections for other testings of the criteria.

♦

Ellipses. Counterexample to the weak criterion.

We continue following the history of mathematics and now consider the family of ellipses $(\mathcal{E}_{a,b})$ of strictly positive semi-axis a and b with $a > b$, generalizing the preceding family of circles $(\mathcal{C}_r)$. In the literature, the parameter a is often called the major semi-axis and the parameter b the minor semi-axis of the ellipse $(\mathcal{E}_{a,b})$.

An important geometric remark is to notice that this family of ellipses can be considered as a "perturbation" of the preceding family of circles $(\mathcal{C}_r)$; or inversely and more significatively, the family of circles $(\mathcal{C}_r)$ can be thought as a "degenerate" subset of the ellipses family $(\mathcal{E}_{a,b})$ corresponding to equal semi-axis; $a = b := r$.

Note also that the modern theory of moduli of algebraic varieties merges those two families into one geometrical object of the same nature, namely an algebraic variety fibered into those elementary conics.

By splitting the parameter space from a unidimensional one of the $r \in \mathbb{R}$ into a bidimensional one of the $(a, b) \in \mathbb{R}^2$, the complexity of the fundamental periods is drastically affected, as we will see in a few moments : this phenomenon is a critical instance of how the fundamental periods encodes the underlying symetries of a variety; here the degenerate cases of highly symetrical varieties, namely the circles $(\mathcal{C}_r)$ are detected within the family of ellipses $(\mathcal{E}_{a,b})$, by the complexity drop of the fundamental periods tuple.

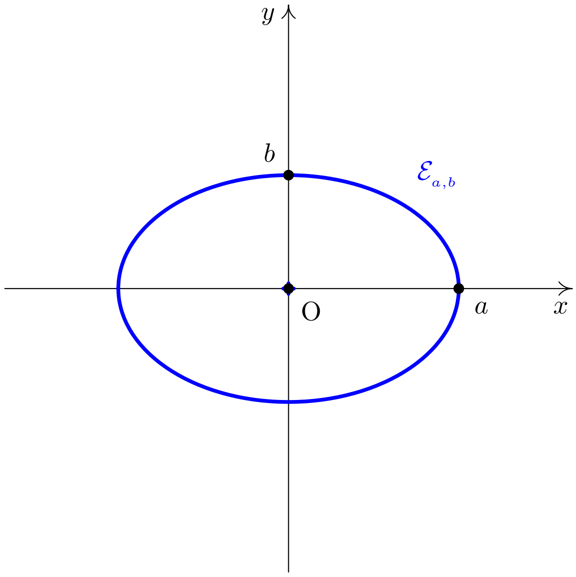


Figure 2.4: Ellipses $(\mathcal{E}_{a,b}) : \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1.$

The mean points of those ellipses are all the origin, so that the conditions of the criteria are therefore fulfilled. For the fundamental arithmetic periods, we get the following classic result.

$$\mu(\mathcal{E}_{a,b}) = \mu(a,b) = \pi ab \quad \text{then}$$
$$\lambda(\mathcal{E}_{a,b}) = \lambda(a,b) = 4aE\left(\sqrt{1-(\tfrac{b}{a})^2}\right).$$

In the expression above, E denotes; in accordance with common conventions, the complete elliptic integral of the second kind, defined by the formula below.

$$\mathbf{E}(k) = \int_0^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2 t}\,dt \quad \text{with} \quad |k| < 1.$$

In terms of transcendental degrees, there is no significant birational simplification of the fundamental arithmetic periods tuple anymore as the one that occured for the previous degenerate case of the family of circles. We only have the following expression that lacks symetry from the second more complicated period :

$$trdeg_{\mathbb{Q}}\mathbb{Q}(\mathcal{P}_{\mathcal{A}}(\mathcal{E}_{a,b})) = trdeg_{\mathbb{Q}}\mathbb{Q}\left(\pi ab,\ aE\left(\sqrt{1-(\tfrac{b}{a})^2}\right)\right).$$

Suppose this transcendental degree maximal, then both fundamental arithmetic periods are transcendental; in particular, the first one gives the transcendence of πab, in other words $ab \notin \frac{1}{\pi}\bar{\mathbb{Q}}$. The transcendence of the other period does not seem to be exploitable in this form lacking of symetry. Let us move on, and come back later, this hurdle might be overcome by sharpening and polishing the criteria formulation in future extended versions.

Suppose conversely that a and b are rational numbers, then $(\mathcal{E}_{a,\,b})$ admits an infinity of rational points by the trivial linear diagonal transport

$$\begin{aligned} H_{a,\,b} \quad : \quad & \mathcal{C}_1 \longrightarrow \mathcal{E}_{a,\,b} \\ & (x,\,y) \longmapsto (ax,\,by) \end{aligned}$$

of those of the trigonometric unit circle $\mathcal{C}_1$; namely $P_t = (\frac{2t}{1+t^2}, \frac{t^2-1}{1+t^2})$ with $t \in \mathbb{Q}$, that give pythagorean triples after homogeneisation; and the following points of the ellipse $(\mathcal{E}_{a,b})$:

$$Q_t = H_{a,\,b}(P_t) = (\frac{2t}{1+t^2}a, \frac{t^2-1}{1+t^2}b)\,, t \in \mathbb{Q}.$$

For example, we get $Q_2 = (\frac{4a}{5}\,, \frac{3b}{5})$ or the symetric « twist » of this point with respect to the diagonal $Q_3 = (\frac{3a}{5}\,, \frac{4b}{5})$.

After multiplicative simplification by rational numbers of the fundamental arithmetic periods, the algebraic nature of $\delta(\mathcal{E}_{a,\,b}) = \delta_{a,b} = \frac{\mu(a,b)}{\lambda(a,b)}$ is identical to that of $\frac{1}{\pi}E(\sqrt{1-(\frac{b}{a})^2})$. In other words, we have the following equality :

$$trdeg_{\mathbb{Q}}\mathbb{Q}(\delta_{a,b}) = trdeg_{\mathbb{Q}}\mathbb{Q}\left(\frac{1}{\pi}E(\sqrt{1-(\frac{b}{a})^2})\right).$$

However, this quotient of classical periods numbers $\frac{1}{\pi}E(\sqrt{1-(\frac{b}{a})^2})$ is notoriously known to be transcendental provided that the parameters a and b are algebraic; see for example Alan Baker book [[Bak90], p.63] or the russian encyclopædia more recent booklet devoted to Transcendental Number Theory written by Yuri Nesterenko and Naim Ilych Feldmann [[PS98b], p.154, thm 3.13] and edited by Alexei Parshin and Igor Shafarevich in the late 90's. We therefore have the first counterexample of the version of the weak criterion given here.

Notice also how this weak criterion applied well previously for the degenerate cases of circles, while collapsing for the non degenerate cases constituing the generic or general situation.

If a and b are rational, they are algebraic and the algebraic independence over $\mathbb{Q}$ of the two fundamental arithmetic periods $\lambda(a,b)$ and $\mu(a,b)$ is equivalent to that of π and $E\Big(\sqrt{1-(\frac{b}{a})^2}\Big)$; in other words

$$trdeg_{\mathbb{Q}}\mathbb{Q}(\mathcal{P}_{\mathcal{A}}(\mathcal{E}_{a,b})) = trdeg_{\mathbb{Q}}\mathbb{Q}\left(\pi,\ E\left(\sqrt{1-(\tfrac{b}{a})^2}\right)\right).$$

Bearing in mind the presence of non-trivial rational points when the parameters a and b are rational numbers, the present version of the stable criterion given in the previous section suggests the algebraic dependence of these two numbers, which to best of our knowledge remains open; but following the known results in Transcendental Number Theory; those complex numbers are expected to be algebraically independent; fitting the "usual" or "general" situation; because, as we already saw in the case of circles, being algebraically dependent for a pair of complex numbers when one of them is a fixed transcendental number, constitutes the exceptional rare situation; those exceptional cases are actually countable.

2.2.2 Synthesis and perspectives.

So, the version given here of the stable criterion quickly seems to need a consolidation or a reformulation; as those lines are written, this process is on the way; going along with a recompilation of the author past research paper notes : the idea emerging so far from that archive excavation is that TWO PERIODS ARE NOT ENOUGH to get robust arithmetical criteria; the first consolidation stage giving the most promising approach, is to consider not only two periods of the studied variety but an array or a matrix of those periods : in the preceding case of ellipses, only one more period may be enough.

The underlying philosophy being that GENERALIZED PERIODS OF A VARIETY TOTALLY DETERMINE ITS ARITHMETIC, and actually and even more generally, GENERALIZED PERIODS OF A VARIETY V TOTALLY CHARACTERIZE the variety V : its arithmetic; its geometry; its symmetries; even its intrinsic nature, that is the class of varieties it belongs to.

The vague formulation of that paradigm is that THE GENERALIZED PERIODS FUNCTOR FROM THE CATEGORY OF VARIETIES TO THE CATEGORY OF GENERALIZED PERIODS SPACES OR MODULAR SPACES OF GENERALIZED PERIODS IS "INJECTIVE" OR "FAITHFUL". This elusive paradigm should find a more precise formulation in the clarifying context of motives; the assertion may stand still when replacing "varieties" by "motives" : the (GPC) or Grothendieck Periods Conjecture going along that way, in terms of injectivity of a rather sophisticated motivic periods evaluation map; we refer the reader to the recent reference book on periods of Nori motives by the bavarian arithmetic geometry working group lead by Annette Huber and Stephan Müller-Stach [HFvWMS18].

To give some rough ideas about that trend of thoughts; the reader may think of the generalized periods process as being able to encapsulate some arithmetic-flavored "DNA" data when picking fundamental arithmetic periods; among a very rich amount of various other data-types or data-flavors when keeping the whole generalized periods space. Actually, those generalized periods store core-data of the studied varieties into complex, mostly transcendental numbers; and bearing that in mind; the reader might consider the reverse process of finding out the relations between those generalized periods as one deployment or one expression of that encapsulated DNA data : when restricting the generalized periods to the arithmetic fundamental ones and the relations governing them to the algebraic ones, we extract the arithmetic-flavored DNA data from the whole DNA data set.

The Betti de-Rham pairing giving most of those fundamental arithmetic periods that concentrate this profusion of information may be thought as the merging of two half-helices of the "DNA" of the variety; in the form of two complexes, namely the topological Betti one and the differential De-Rham one.

Analogously to the biological proteins synthesis, the structural expressions of that encapsulated DNA data gives all the characterizing "shapes" of the variety : its nature or "identity face" signature, with its distinctive features like its symmetries; its arithmetics or its "arithmetic shape"; so most of all its "structural shapes".

In other words, according to the nature of the chosen generalized periods, we get the different "genes" or genotypes of the varieties; the underlying structures or traces of patterns emerging from the relations governing those chosen generalized periods, finally reveal out or generate the corresponding different "phenotypes" or phenotypic expressions of those genotypes.

But enough biological digressions, although there could be a lot to uncover from the interplay of deep analysis of biological phenomena and mathematics in the footsteps of René Thom investigations; let us conclude before getting back to hardcore bone dry mathematics and continue our experiments :

The main ontological issue for those arithmetic criteria will be the choice of generalized periods that will carry the qualifier "fundamental arithmetic periods".

For the present versions of the criteria given here, the fundamental arithmetic periods are constituted of the two periods coming from fundamental Betti classes, namely the two periods

μ and λ; aside of the generalized periods of the mean point. As we saw, those two chosen ones μ and λ so far, will have to be completed in order to get the right fundamental arithmetic tuple, since those only two periods are not enough even for the quite simple cases of conics defined over $\mathbb{Q}$ that are among the most elementary algebraic plane curves defined over $\mathbb{Q}$.

What appears from the recompilation of the author past investigations so far, is that the wider the target class of varieties is, the more numerous are generalized periods needed to constitute a robust fundamental arithmetic tuple, in other words; the wider should be the needed tuple of fundamental arithmetic periods.

Here, the coined term "qualitative space of arithmetic criteria" in this memoire introduction (1.1 p.9), facing the contradicting-varieties strata, finally shows-up. The main advantage of our proposed criteria is that the corresponding "qualitative space" is directly uniformized by the choice of generalized periods that will be called *fundamental arithmetic periods*. We can parallel this choice to the Group choice of the torsors occuring in the algebraic cases of the Brauer-Manin theory.

If N is the number of generalized periods chosen for the fundamental arithmetical tuple denoted by $\mathcal{P}_{\mathcal{A}_N}(V) = (p_1(V), \ldots, p_N(V))$, so that we form an increasing serie, meaning that we successively add one period to pass from one tuple to the next one; we clearly get a filtration of varieties satisfying the corresponding criteria of absence of non-trivial rational points, namely

$$(\mathfrak{C}_N): \qquad trdeg_{\mathbb{Q}}\mathbb{Q}(\mathcal{P}_{\mathcal{A}_N}(V)) = N \qquad \Rightarrow \qquad \dot{V}(\mathbb{Q}) = \emptyset.$$

The second main advantage of our approach is another uniformization process occuring this time; at the lower level of those chosen generalized periods, in the case of *algebraic varieties defined over* $\mathbb{Q}$: this low-level uniformization emerges from the expression of generalized periods as values of a SINGLE type of complex functions, namely hypergeometric functions; with this process being potentially able to hatch a still embryonary "Hypergeometry" closely related to the recently new born "Motivic geometry". To give a rough idea about the consequences of this phenomenon, the reader might think of the modularity of elliptic curves defined over $\mathbb{Q}$.

In this setting of *algebraic varieties defined over* $\mathbb{Q}$, the preceding criteria stratification of varieties has a corresponding functional side through the hypergeometric functions spaces ${}_p\mathcal{H}_q(\mathbb{Q})$ with $p+q = f(N)$, f being an increasing integer function of N; those spaces are subspaces of the following space of hypergeometric complex functions with algebraic parameters, namely :

$${}_p\mathcal{H}_q(\bar{\mathbb{Q}}) = \{\, {}_pF_q(\mathbf{a}, \mathbf{b}; \phi(z)) |\ \mathbf{a} \in \bar{\mathbb{Q}}^p,\ \mathbf{b} \in \bar{\mathbb{Q}}^q,\ \phi \in \overline{\mathbb{Q}(z)}\}.$$

This thread of ideas will hopefully be precised in forecoming versions of this memoire or elsewhere, they are; till now, quite foggy, but to give a hint to the curious reader, a motive associated to the variety should be embodied by or related to the arguments spaces [2] of the preceding hypergeometric expressions and the algebraic dependence relations of those functions values, that matter for our criteria; should emerge from "twisted symmetries" in an Hopf and Lie Algebras context within the structure of those arguments spaces, probably encoded by a modular flavored motivic Galois group associated to the variety, this group should also encode by its splittings sizes drops, the different "depths" of this low-level hypergeometric uniformization.

The first depth being the number of packets of fundamental arithmetic periods expressible by the same (p, q), the other depths being related to each one of those packets and given by the number of fundamental arithmetic periods expressible with the same parameters subcomponents (a_i, b_j) of $(\mathbf{a}, \mathbf{b})$, going along a filtration of the functions spaces ${}_p\mathcal{H}_q(\mathbb{Q})$.

Here the reader should think of these splittings of the set of fundamental arithmetic periods and the corresponding hypergeometric functions spaces in terms of splittings of the associated motive and its modular flavored motivic Galois group, a reminiscence of the spliting filtration of the Galois group and the associated fields extensions of classical Galois theory.

Some references coming to mind are surprinsingly from Mathematical Physics with the theory of Group representation applied to special functions. Names to look for are Naum-Jakoblevich Vilenkin [Vil68] from the red army side of USSR and Willard Miller for the uncle Sam side, the latter has published an article on Lie Groups and Hypergeometric functions in the cold war mid 1960s [Mil68]. What comes out of those references for our concern is that classical linear algebraic groups like $GL_{p+q}(\mathbb{Q})$ together with its subgroups such as $SL_{p+q}(\mathbb{Q})$ for the parameters $(\mathbf{a}, \mathbf{b})$ on one side and $SL_2(\mathbb{Q})$ for the complex variable z on the other side, should play a major role in the preceding modular motivic scene : the main interest of this hypergeometric uniformization process is to provide a concrete grasp on the rather "fugace" motives and the Motivic Galois Groups through specific products of Classical Linear Algebraic Groups. As a relevant example of a motivic upgrade of the preceding 1960s theories is a recent result close to all that uniformization trend of thoughts and found in the $(2, 3)$-theorem of Francis Brown [Bro12] on multizetas values, previously Mike Hoffman conjecture and proved around 2010, stating that those values are $\mathbb{Q}-$linearly generated by the values from the tuples consisting only in 2s and 3s : this result should be closely related to the hypergeometric motivic framework coined in the term "embryonary Hypergeometry".

One of the points that remains particularly foggy to the author for our concern, is weither or not, a strong uniformization is reachable for all *standard varieties*, meaning that we could expect $p + q \leq C_{\mathfrak{X}}$ for some small integer constants $C_{\mathfrak{X}}$; depending only on the intrinsic nature,

[2] The general philosophy is that Hypergeometric data are motives manifestation or incarnation.

type or complexity $\mathfrak{T}$ of the studied classes of varieties : the constant should increase with $\mathfrak{T}$; for instance, it should be greater for transcendental varieties than algebraic ones; going along with the required enlargement of the tuple of fundamental arithmetic periods needed for the targeted classes.

It should also depend of the chosen range for the variable of the hypergeometric functions : in the general case, we might be forced to consider algebraic values for the range of the variable z of the hypergeometric functions in the form of $\phi(z)$. If we want to restrict the range of the variable z of the hypergeometric functions only to rational numbers, we might have to pay the price of inflating the uniformization upper bound $C_{\mathfrak{T}}$. Another way of keeping this upper bound as low as possible may consist in shifting the functional side from the one variable Hypergeometric functions spaces to their multivariable extensions, namely the Appell-Lauricella hypergeometric functions spaces, or their modern generalizations, the GKZ-hypergeometric functions spaces, that seem to be the most appropriate for a "motivisation" process, fitting the trend of thoughts introduced here. Indeed, Guelfand-Kapranov-Zelevinsky hypergeometric treatment [GKZ90] is a systematic clarifying discrete geometrisation of the rather foisonous theory of hypergeometric functions through the introduction of parametrizing spaces as $\mathbb{Z}$-lattices of $\mathbb{C}^n$; with that in mind, the above $p+q$ correspond to the size of the paramatrizing spaces.

What the reader should keep from thee preceding quite elusive elucubrations of the author, is

1) The universality of the matriochka of the multivariable series spaces $\{\mathbb{Q}[[z_0, ..., z_n]],\ n \in \mathbb{N}\}$ and specifically, its hypergeometric subspaces ${}_p\mathcal{H}_q(\mathbb{Q})$ that are the bulb roots of "Hypergeometry".

2) The key role of Group Theory through classical Linear Algebraic Groups applied to the rational parameters spaces $\mathbb{Q}^p \times \mathbb{Q}^q$ of the hypergeometric parametrized subspaces ${}_p\mathcal{H}_q(\mathbb{Q})$, for Number Theory and Arithmetic questions, this is the motivic Galois groups playground, appropriate for solving a bunch of diophantine additive problems (Perfect numbers through dihedral groups, $A = B + C$ Fermat type equations, etc).

3) The motivic process of uniformization consisting in restricting as much as possible the hypergeoemetric functions spaces through the size $p+q$ of their rational parameters spaces, while keeping the sought universality for the targeted classes of varieties.

A recent tangible motivic result widening the first point 1) is the proof by Joseph Ayoub [Ayo15] around 2015 of a relative (namely multivariable functional) version of the Kontsevich-Zagier periods conjecture, the latter being quasi-equivalent to the forementioned Grothendieck periods conjecture (GPC).

As a conclusion, that does not end thoughts but rather opens perspectives about the coined "universality", the meditations of the author suggest that as periods numbers are the intermediate link allowing motives "immixtion" between algebraic numbers and a large part of transcendental numbers; the hypergeometric functions that are functional genitors of periods, constitute the corresponding intermediate link between algebraic functions and a large part of transcendental ones, allowing also motives "immixtion" between the two classes : those special functions embody a common unification into the *standard varieties* realm of its two subrealms, namely the algebraic varieties and the *geometric* transcendental ones, through their underlying $\mathbb{Q}-$motives structuring ability.

This is "Hypergeometry", a geometry above classical ones : its linking ability comes from the fact that Hypergeometric varieties are quasi-stable under the periods map; here an hypergeometric variety is taken to be either the zero locus of multivariable hypergeometric functions or a manifold parametrized by such hypergeometric functions. The preceding uniformization process of point 3) consists in restricting as much as possible the intermediate hypergeometric functions spaces while keeping the sought stabilisation of the periods map on the targeted classes.

Indeed the periods map generally sends a parametrized family of algebraic varieties over $\mathbb{Q}$

$$\mathcal{V} = (V_{t_1 \dots t_n})_{(t_1 \dots t_n) \in \mathbb{Q}^n}$$

into a transcendental variety

$$\mathcal{P}(\mathcal{V}) = \{(p_1(V_{t_1 \dots t_n}), \dots, p_N(V_{t_1 \dots t_n})) \,|\, t_1 \in \mathbb{C}\} := \mathcal{W}.$$

Notice that $\mathcal{W} = (W_{t_2 \dots t_n})_{(t_2 \dots t_n) \in \mathbb{Q}^{n-1}}$ so that the periods of this transcendental variety $\mathcal{W} \subset \mathbb{C}^N$ gives a parametrized space of higher periods, the size of the parametrizing space drops down by one at each iteration of the period map while the intermediate level of the periods increases by one and the process might be continued till the exhaustion of the initial parametrizing space, reaching the terminating level n : the terminal stage finally gives just a simple tuple of higher periods numbers.

Some of those successive terminal periods are caught by successive higher hypergeometric function spaces, meaning with higher $p + q$, and this iterating process generates a filtration of those special higher periods numbers, by the level. Classical or effective periods of Kontsevich-Zagier like the notorious $\pi = 2\ {}_2F_1(1, 1; \frac{3}{2} \,|\frac{1}{2})$ are level-0 periods, whereas its inverse $\frac{1}{\pi}$ might not be a classical or level-0 period but rather a higher level period, as suggested by the hypergeometric nature of some remarkable formulas by Srinivasa Ramanujan for this number, for instance, among hundreds of formulae : $\frac{1}{\pi} = \frac{1}{4}\ {}_4F_3(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{7}{6}; \frac{1}{6}, 1, 1 \,|\frac{1}{4})$. In this setting, algebraic numbers might be considered as "periods of level -1".

Now, the key question is : HOW TO CATCH THE GRAIL OF THE REPRESENTATION OF PERIODS BY HYPERGEOMETRIC FUNCTIONS? There seem to be at least three ways of achieving that goal at least in the case of algebraic varieties defined over $\mathbb{Q}$:

a) Compute directly the periods of the studied variety and figure out during the computations, how to express them in hypergeometric terms. Hard way but tractable, by the integration of differential forms with coefficients in rational powers of rational functions or Laurent polynomials, giving Eulerian integrals; that have an hypergeometric essence.

b) Use indirectly Ordinary Differential Equation (ODE) techniques : dip the studied variety V into a parametrized family $(V_t)_t$, find the ODE satisfied by the corresponding parametrized periods and look for the space of hypergeometric functions satisfying the closest ODE. This dynamisation process amounts to a category shift to the Galois theory of differential equations and operators : standard equations involving variables are substituted by differential equations and operators involving functions.

As the name Galois pops up here, the two theories may actually share a common motivic backbone. One of the key theorems linking these two theories is Siegel-Shidlovsky theorem on E-functions : an hypergeometric version of this theorem initially dealing only with E-functions, would actually linearize the Grothendieck Periods Conjecture, giving on the go, an estimate of the size of motivic Galois groups in terms of ranks of sheaves on hypergeometric D-modules. A few words about that may be found in the appendix 3.

c) The straightforward bold way : push even further the uniformization process to ... the studied varieties themselves, by getting at least local if not global parametrizations of them involving hypergeometric functions, then invoke the quasi-stability of the periods map.

In short terms every *standard variety* V should be at worst a twist of an hypergeometric variety by a *geometric* transcendental one; so that its periods $p(V)$ are given by the product of the value of an algebraic function R at *higher* transcendental values by the value of another one S at hypergeometric values :

$$p(V) = R(\tau_1(u_1), \ldots, \tau_n(u_n))\; S(H_1(z_1), \ldots, H_n(z_n)). \tag{2.2}$$

The *higher* transcendental values $\tau_i(u_i)$ might not be expressible in hypergeometric terms, so that if V is hypergeometric then its periods variety is not fully but only almost hypergeometric, this is the quasi-stability of the periods map on hypergeometric varieties. At that stage, two cases show-up, corroborating the general principle that periods of a variety are able to capture its intrinsic nature.

1) If V is algebraic, the *higher* transcendental values $\tau_i(u_i)$ may be uniformized by the Euler

gamma function as $\tau_i(u_i) = \Gamma(u_i)$, increasing the depth of uniformization :

$$p(V) = R(\Gamma(u_1), \ldots, \Gamma(u_n))\; S(H_1(z_1), \ldots, H_n(z_n)). \tag{2.3}$$

2) If moreover V is an *algebraic variety defined over* $\mathbb{Q}$; the gamma values may be values at rational numbers so that they might be already expressible in hypergeometric terms, namely as values of algebraic functions at hypergeometric values:

$$\Gamma(u_j) = \phi_j(H_1(s_1), \ldots, H_n(s_n)). \tag{2.4}$$

Now, invoke the potential uniformization of algebraic functions by hypergeometric ones to finally close the uniformization loop into a full hypergeometric one in this quite peculiar case of *algebraic varieties defined over* $\mathbb{Q}$. So, ultimately in this case, the transcendental degree of the periods of the studied variety is given by the transcendental degree of the hypergeometric values corresponding to those periods. For further developments, the reader is invited to read the appendix (3).

The reader should also note that the point 2) is the manifestation of a critical motivic convergence phenomenom that partly unveils the Unity of standard mathematics : suppose given a higher transcendental function that is arithmetic, meaning built from Number Theory (L-functions and Zeta functions from counting, Gamma interpolating the factorial, etc) then if it is not globally expressible in hypergeometric terms; its values at rational numbers might have hypergeometric expressions. An instance of this trend of thoughts is Zagier conjecture relating the values of Dedekind Zeta functions at integers to polylogarithms : indeed, it turns out that those polylogarithms admit highly uniformized hypergeometric expressions.

That unclear babblings captured from the Hypergeometry embryo obviously need to be corrected and translated into an intelligible rigorous mathematical langage, chances are meager that this will be done by the author for trivial material reasons, but rather left to future generations. Now we shall move on to concrete reality and continue our down-to-earth testing of the two periods versions of the criteria on some other elementary varieties.

Normally, following history, after those elementary conics, we should have gone-up one degree, investigating some cases of cubics. Since the original french text skipped them, we will also skip those temporarily but the reader should be informed that the few recently recompiled cases from past research paper notes comprise the loop components of some nodal elliptic curves defined over $\mathbb{Q}$: surprisingly, even the present version of the stable criterion given here applied quite well for all the nodal pieces of those curves that were tested, confirming hopes for the proposed criteria.

Indeed, a polishing of their formulations, may make them as quite powerful investigating tools for the study of rational points with one of the widest spectrum, applying from finite sets on one side of the spectrum, to some very general topological varieties on the far away opposite side, passing through traditional algebraic varieties and the less explored transcendental ones.

In order to moderate this surely excessive optimistic view, let us again get back to some quite elementary concrete cases.

2.2.3 Other elementary algebraic cases : some plane quartics.

Still following the historical development of mathematics and going-up again one degree, quartics curves are also among the first ever algebraic varieties studied by mankind; those are 1-dimensional algebraic varieties of degree 4 embedded in a plane; or plane curves given as the 0-locus of degree 4 polynomials in 2 variables.

As a side note, lurking towards future geometrical classification objects like cohomological or discrete invariants; for plane algebraic curves, the degree has emerged as the first obvious classification gauge; having in mind that for a smooth projective curve (C), the topological one $g(C)$ of the genus as another classifying parameter is intimately linked to the former one of the degree $d(C)$ by the classical degree-genus formula,

$$g(C) = Q(d(C))$$

where Q is a quadratic or degree-2 polynomial with rational coefficients.

Cardioids.

In polar coordinates, a being a strictly positive real parameter, those curves are given by the following equation.

$$(C_a) : \rho(\theta) = a(1 + \cos\theta).$$

In cartesian coordinates, they are given by the equation in the caption of the pictures that represent them; giving them again the status of algebraic curves. Their fundamental arithmetic periods μ_a and λ_a are easily computed using the polar form of their defining equation; we obtain

$$\mu_a = \mu(C_a) = \frac{3}{4}\pi a^2 \quad \text{and} \quad \lambda_a = \lambda(C_a) = 8a.$$

Still using the polar form, we determine easily their mean points which are written as follows

$$G_a = \big(\frac{4}{5}\, a, 0\big).$$

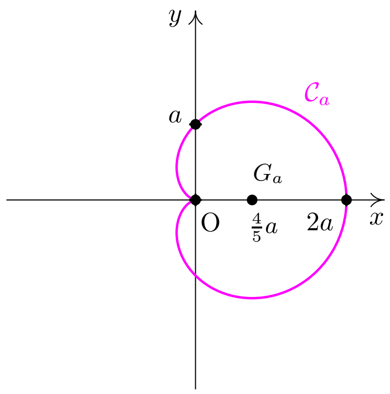


Figure 2.5: Cardioids $(C_a) : (x^2 + y^2 - ax)^2 = a^2(x^2 + y^2)$.

The fundamental arithmetic periods pair $\mathcal{P}_{\mathcal{A}}(C_a)$ is equivalent in terms of transcendental degree to the pair (a, π), in other words

$$trdeg_{\mathbb{Q}}\mathbb{Q}\left(\mathcal{P}_{\mathcal{A}}(C_a)\right) = trdeg_{\mathbb{Q}}\mathbb{Q}\left(\pi, a\right).$$

Suppose this degree is maximal or equivalently that the two fundamental arithmetic periods are algebraically independent, then a is transcendental and therefore, bearing in mind the algebraic equation of this curve, the only possible rational point is the trivial singular point of the origin; so that the curve has no non-trivial rational point.

Inversely, suppose that the curve admit a non-trivial rational point, then still from its equation; a is necessarily algebraic; in this case the mean point is also algebraic and we have the algebraic dependence of the two fundamental arithmetic periods; since the second period is algebraic : it is a degenerate case that fulfills the rare exception of getting a period belonging to $\bar{\mathbb{Q}}$, that is a period that ends up being an algebraic number; giving a degenerate algebraic dependency of the fundamental arithmetic periods. The stable criterion applies particularly well here.

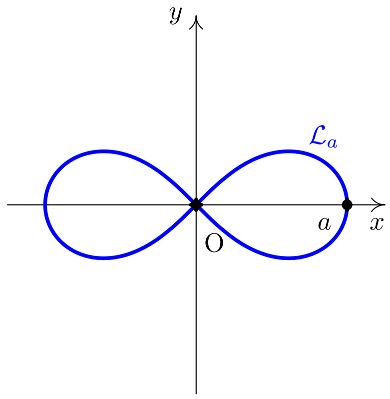


Figure 2.6: Lemniscates $(\mathcal{L}_a) : (x^2+y^2)^2 = a^2(x^2-y^2)$.

Lemniscates.

In polar coordinates, a still being a real parameter strictly positive as for cardioids, their equations are given by the following expression

$$(\mathcal{L}_a) : \rho^2(\theta) = a^2 \cos 2\theta.$$

In cartesian coordinates, they are defined by the equation in the caption of the above graph representing them; also confering them a status of algebraic curves. Their first periods μ_a, can easily be computed with their polar form and are expressible as follows.

$$\mu_a = \mu(\mathcal{L}_a) = a^2.$$

Their second periods, λ_a require more effort —see, for example, Derek Lawden's book [[Law89], p.103] and are obtained as elliptical integrals, as follows

$$\lambda_a = \lambda(\mathcal{L}_a) = \frac{a}{\sqrt{2\pi}}\,\Gamma^2(\frac{1}{4}).$$

The transcendental degree of the field extension generated by the two fundamental arithmetic periods is equal to the one of the field extension generated by the pair $(a, \frac{\Gamma(\frac{1}{4})}{\pi})$:

$$trdeg_{\mathbb{Q}}\mathbb{Q}(\mathcal{P}_{\mathcal{A}}(\mathcal{L}_a)) = trdeg_{\mathbb{Q}}\mathbb{Q}(a, \frac{\Gamma(\frac{1}{4})}{\pi}).$$

Suppose the fundamental arithmetic periods algebraically independent, then a is transcendental; so that if $(\mathcal{L}_a)$ happens to admit a rational point, the algebraic equation of the curve forces this point to be the trivial origin; in other words, this curve has no non-trivial rational points.

Inversely, suppose that the curve admits a non-trivial rational point, then by the same equation argument, this implies that a is algebraic and more precisely that a^2 is rational; so that the first period is a rational number; giving a totally degenerate case of algebraic dependency of the fundamental arithmetic periods. Once again, the stable criterion applies particularly well here, conforting its future developments.

2.2.4 General plane curves : both algebraic and transcendental plane curves.

Here the memoir underlying goal emerges for the first time : extending the study of rational points from the traditional framework of algebraic varieties to the much less known one of non-algebraic varieties; and treating both cases at the same time with one type of criteria : to the best of our knowledge, this has never been attempted so far; the ultimate goal being to extend even more the target range of the criteria candidates to varieties that may not be given by an equation; like boundaries of convex bodies or compact connected subspaces of the real euclidean space and other topological varieties as tackled by the generalized version of the criterion illustrated by the figure 2.2p27; in an attempt of ≪ lier "le Nombre" et "la Forme" ≫ or linking "the Number" to "the Shape"; in the footsteps of Grothendieck "philosophy" or "Jugentraum" of bridging together Topology and Number Theory, hence the chosen terms "general varieties" in the title of this memoir. For that purpose, the differential geometry framework used for periods evaluation may be dropped for the more general one of Geometric Measure Theory.

Fermat curves.

We are going to study the famous Fermat plane curves which led to write all this (see the rocambolesque appendix 3.2 p 68).

We know after the important work of Andrew Wiles and the mathematical community that was contemporary or not; that for an odd prime integer p, these curves do not admit any non-trivial rational point. The stable criterion has the power to treat more general Fermat curves, since it allows to consider the case where the parameter p is no longer an integer, extending thus

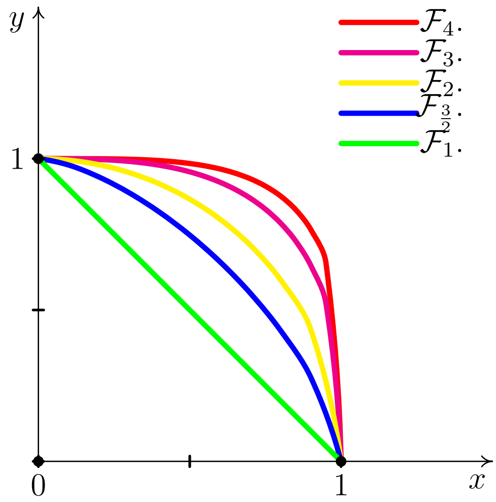


Figure 2.7: Fermat curves $(\mathcal{F}_p) : x^p + y^p = 1$.

the field of investigation to non-algebraic cases. After denoting $\alpha_p = \frac{1}{p} - 1$, long computations of integrals of differential forms make it possible to obtain the periods — the second has not been seen in the abundant literature reviewed — of these curves, giving the values

$$\mu(\mathcal{F}_p) = \frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{p}\right)}{p\,4^{\alpha_p}\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)} \qquad \text{and} \qquad \lambda(\mathcal{F}_p) = \frac{4}{p}\int_0^1 \sqrt{(1-u)^{2\alpha_p} + u^{2\alpha_p}}\; du.$$

The stable criterion would fit that case if we had these two quantities algebraically independent which would amount to have, at least for a strictly positive rational number p, the algebraic independence of $\frac{\sqrt{\pi}\,\Gamma(\frac{1}{p})}{\Gamma(\frac{1}{p}+\frac{1}{2})}$ and $\int_0^1 \sqrt{(1-u)^{2\alpha_p} + u^{2\alpha_p}}\; du$, in other words and transcendental terms

$$trdeg_{\mathbb{Q}}\mathbb{Q}\left(\mathcal{P}_{\mathcal{A}}(\mathcal{F}_p)\right) = trdeg_{\mathbb{Q}}\mathbb{Q}\left(\frac{\sqrt{\pi}\,\Gamma(\frac{1}{p})}{\Gamma(\frac{1}{p}+\frac{1}{2})}, \int_0^1 \sqrt{(1-u)^{2\alpha_p} + u^{2\alpha_p}}\; du\right).$$

This algebraic independence would imply the absence of non-trivial rational points on general Fermat type curves : a teasing hint to the curious reader is that the famous conjecture in Transcendental Number Theory of the algebraic independence of π and values at odd integers of the Riemann ζ function implies right away Fermat theorem through the present criteria; showing-up the quite powerful aspect of Grothendieck Periods Conjecture, often quoted as the (GPC) conjecture : this celebrated motivic conjecture would actually sweep-out instantly; through the criteria exposed here, a vast amount of interrogations on the presence of rational points on general varieties. Again, about the (GPC) conjecture, we refer to the bavarian book on periods of Nori Motives[HFvWMS18].

Now, come the most interesting examples of this short memoir : we treat some cases of general differentiable varieties, covering both algebraic and transcendental ones; the first stage suited framework should ideally be complex analytic varieties when considering the extensions of the field of definition or base field of those varieties to the well known and fruitful field of complex numbers $\mathbb{C}$ possibly getting a compactification or an analytic uniformization process on the go; the second far away framework would be to drop the need of an equation and to tackle some classes of topological varieties like boundaries of "convex bodies" or compact connected subspaces of the real ambient euclidean space.

Some general transcendental plane curves.

We consider the graphs $\mathcal{G}_n$ for an integer $n \geq 1$; symmetrized about the coordinates axis, of the $n-$th powers of the elementary cosine function denoted g_n and restricted to the interval $[0, \frac{\pi}{2}]$.

The purpose of the symmetrization process about the coordinates axis is to trivialize the mean point, this gluing about coordinates axis of pieces of transcendental curves being an instance of the extension of rational points investigations to the boundary of convex bodies or compact connected subspaces of the ambiant euclidean space.

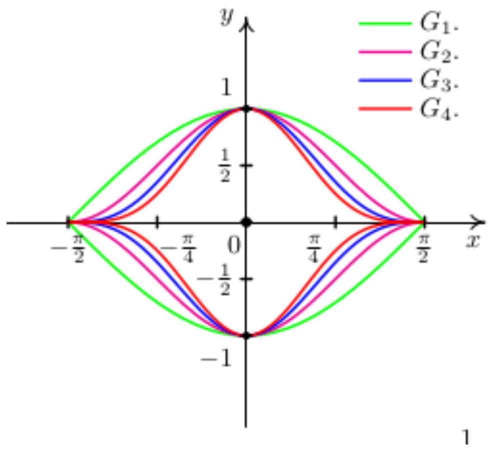


Figure 2.8: Symmetrized Cosine powers $(G_n) : y = \cos^n x$.

The first period is the quadruple of the classic Wallis integral and is given by the following expression

$$\mu(G_n) = 4\int_0^{\frac{\pi}{2}} \cos^n x\, dx = 4\,\pi\frac{\Gamma\,(n+1)}{2^{n+1}\Gamma^{\,2}(\frac{n}{2}+1)}.$$

On the other hand, the second period; escapes again, despite a decent amount of efforts, any simple evaluation, we just have the following expression.

$$\lambda(G_n) = 4\int_0^{\frac{\pi}{2}} \sqrt{1+n^2\sin^2 t\cos^{2(n-1)} t}\, dt.$$

Those expressions give the following transcendantal degree of fundamental arithmetic periods.

$$trdeg_{\mathbb{Q}}\mathbb{Q}(\mathcal{P}_{\mathcal{A}}(G_n)) = trdeg_{\mathbb{Q}}\mathbb{Q}\left(\pi\frac{\Gamma\,(n+1)}{2^{n+1}\Gamma^{\,2}(\frac{n}{2}+1)}, \int_0^{\frac{\pi}{2}} \sqrt{1+n^2\sin^2 t\cos^{2(n-1)} t}\, dt\right).$$

So that the stable criterion would fit this case; bearing in mind the absence of non-trivial rational point on those curves, if the two numbers $\mu_n = \pi\frac{\Gamma\,(n+1)}{\Gamma^{\,2}(\frac{n}{2}+1)}$ and $\lambda_n = \int_0^{\frac{\pi}{2}} \sqrt{1+n^2\sin^2 t\cos^{2(n-1)} t}\, dt$ were algebraically independent for $n \geq 1$.

Since the first number μ_n is rational for n odd, this preceding assertion falls apart : here again, we find out that the stable criterion necessary condition is not satisfied; because the curve (G_n) has no non-trivial rational point but its fundamental arithmetic periods are trivially algebraically dependent; so to eventually get such a sufficient and necessary conditions statement, the stable criterion must be refined with another extended version to make it more powerful.

Remains the case when n is even, and after an obvious simplification of the period μ_n, we are lead to ruling out the algebraic independence of π and λ_n, in other words

$$trdeg_{\mathbb{Q}}\mathbb{Q}(\mathcal{P}_{\mathcal{A}}(G_n)) = trdeg_{\mathbb{Q}}\mathbb{Q}\,(\pi, \lambda_n)\,.$$

For example; if $n = 2$, we have $\lambda_2 = \sqrt{2}E(\frac{\sqrt{2}}{2})$, so that λ_2 is congruent to $E(\frac{\sqrt{2}}{2})$ modulo multiplication by a scalar of $\bar{\mathbb{Q}}$, where E is the usual complete elliptic integral of the second kind : we unexpectedly end up with the algebraic independence issue related to the algebraic smooth case of the ellipse $(\mathcal{E}_{\sqrt{2},1})$ belonging to the family of ellipses $(\mathcal{E}_{a,\,b})$ specialized at the particular value of the parameter (a, b) given by $a = \sqrt{2}$ and $b = 1$. This case leads; despite the non rationality of the parameter a, to the trivial rational point $(0, 1)$ and the equation $x^2 + 2y^2 = 2$; together with the question about the nature of the algebraic dependence of π and $E(\frac{\sqrt{2}}{2})$; so that

$$trdeg_{\mathbb{Q}}\mathbb{Q}(\mathcal{P}_{\mathcal{A}}(G_2)) = trdeg_{\mathbb{Q}}\mathbb{Q}\left(\mathcal{P}_{\mathcal{A}}(\mathcal{E}_{\sqrt{2},1})\right) = trdeg_{\mathbb{Q}}\mathbb{Q}\left(\pi, E(\frac{\sqrt{2}}{2})\right).$$

2.2.5 Synthesis.

Bearing in mind that π and $E(\frac{\sqrt{2}}{2})$ are actually algebraically independent; and that this particular ellipse $(\mathcal{E}_{\sqrt{2},1})$ does have many non-trivial rational points; we are again facing the insufficiency of only two periods to get a robust stable criterion; by adding more periods, say at least one more in this particular case; the author is confident about the stable criterion assertion; and is rather optimistic about its future extended versions; a hint for the curious reader about those extensions is to look in the direction of curvature-periods; in other words; periods related to the curvatures of the studied varieties.

A key point to also notice is the unexpected coincidence of this case that might open the road of deep heuristic investigations about the criterion invariance under smooth and singular transformations; since, starting from (G_2), a globally non-smooth enclosed curve that turns out to be a piecewise-smooth transcendental enclosed curve; the same algebraic independence issue of fundamental arithmetic periods of $(\mathcal{E}_{\sqrt{2},1})$, a globally smooth algebraic enclosed curve defined over $\mathbb{Q}$ was encountered, giving the transcendental degree formulation :

$$trdeg_{\mathbb{Q}}\mathbb{Q}(\mathcal{P}_{\mathcal{A}}(G_2)) = trdeg_{\mathbb{Q}}\mathbb{Q}\left(\mathcal{P}_{\mathcal{A}}(\mathcal{E}_{\sqrt{2},1})\right).$$

This particular result reinforces the powerful and profound intuition that Topology with its varieties is the suitable setting for torture-testing Arithmetic criteria by studying their extensions to general topological varieties and seeing how "the arithmetic" is influenced by "the geometrical or physical aspect". This also conforts the belief that the full understanding of rational points on varieties can not skip the fields of investigations of Arithmetic Topology; and particularly, one of its foundational base that is Grothendieck topos theory.

Other facts that confort this point of view are the crucial roles of topological invariants or objects, that in fine; determine the arithmetic of varieties : let us cite the topological genus classification for algebrac curves over number fields of Gerd Faltings theorem mentioned in the first part of the memoir, or the topological invariants intervening in the Brauer groups of real algebraic varieties mentioned in the conclusion of this first part.

In a more prestigious context, let us mention the topological invariants pervading André Weil conjectures both statements and proofs; just to mention one of the numerous celebrated instances of that phenomenon. In the same era, let us mention Georges de Rham isomorphism theorem unveiling the Topological nature of the Differential Cohomology of real differentiable manifolds, that aferwards took this mathematician name to be De Rham cohomology. In a more recent perspective, let us also evoke the recent achievement in the proof of the extended Chabauty theorem by John Coates and Kim Minhyong, using the theory of Arithmetic Fundamental

Groups; as well the latter Kim Minhyong proof of Siegel theorem on the finiteness of integer points on curves; using the theory of Etale Fundamental Groups.

The author is personally convinced that the Topology approach is the most promising direction to take for the study of rational points on varieties; hence the evocated extensions of the criteria given here to some classes of topological spaces. The ultimate theoretical setting for the criteria given here being the already mentioned one of Topos theory, either its foundational logic sides with the classifying topos of first-order geometrical theories for the scrutining of the theoretical bridging contents proposed by those criteria; or its classical sides through its cohomological applications, that encompass the already very promising theory of motives. To conclude, Topos theory appears as the natural setting to give a precise meaning to the coined terms "qualitative Space of arithmetic criteria", a criterion being an element of a suitable Arithmetic Topos.

We can also derive other important heuristic facts from the preceding transcendental equality

$$trdeg_{\mathbb{Q}}\mathbb{Q}(\mathcal{P}_{\mathcal{A}}(G_2)) = trdeg_{\mathbb{Q}}\mathbb{Q}\left(\mathcal{P}_{\mathcal{A}}(\mathcal{E}_{\sqrt{2},1})\right).$$

Indeed, the left-hand side variety, namely (G_2) has only two rational points that are moreover trivial, whereas the right-hand side variety $(\mathcal{E}_{\sqrt{2},1})$ has an infinite number of non-trivial rational points, so that we have another reason for patching the tuple of fundamental arithmetic periods in order to get robust arithmetic discriminants able to treat general varieties.

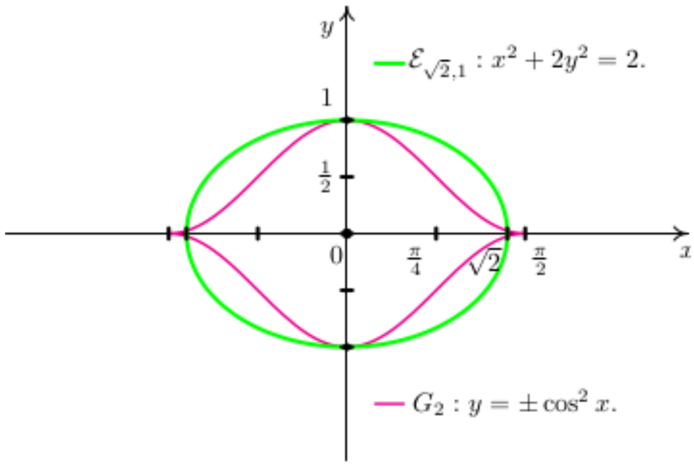


Figure 2.9: Two equiperiodic curves (G_2) and $(\mathcal{E}_{\sqrt{2},1})$.

Another obvious but important derived fact is that having non-trivial rational points is surely not a full topological nor a full homotopical property, since both varieties (G_2) and $(\mathcal{E}_{\sqrt{2},1})$

treated here are homeomorphic and even homotopic to each other, but have totally different arithmetics. Homotopic and homeomorphic equivalences are way too souple compared to the very rigid arithmetic one.

To conclude this section and provide at least another proof or actually a sketch of proof in this memoir, in order to give to the reader something back for the time devoted to reach those lines; here is an outline of a possibly original proof for the asserted algebraic independence of π and $E(\frac{\sqrt{2}}{2})$.

Proposition 5. *The two numbers π and* $\mathrm{E}(\frac{\sqrt{2}}{2})$ *are algebraically independent over $\mathbb{Q}$.*

Proof. The strategy to obtain this result consists in five steps.

1. Remind the Legendre formula that $\mathbb{Q}$-linearly links some products of the elliptic integrals of the first two kinds and their conjugates to π :

$$E'K + EK' - KK' = \frac{\pi}{2}.$$

   Keep in mind that the quote in the above formula does not denote the symbol of the derivative but rather indicates the conjugate value of those elliptic integrals, precisely their values at the conjugate argument or complementary modulus usually denoted k', whence $E = E(k)$, $E' = E(k')$, $K = K(k)$ and $K' = K(k')$, with $k' = \sqrt{1-k^2}$.

2. Take this Legendre relation at the concerned modulus $k = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$, that turns out to be the unique positive normal modulus in the open unit disk $|k| < 1$, "normal" modulus meaning that the modulus k is equal to its conjugate or complementary modulus $k' = \sqrt{1-k^2}$:

$$2EK - K^2 = \frac{\pi}{2}.$$

3. Derive from the previous point the fact that the algebraic independence of (K, π) implies the algebraic independence of (E, π) at this particular value of the modulus or more precisely in terms of transcendental degree :

$$trdeg_{\mathbb{Q}}\mathbb{Q}\,(\pi, E) = trdeg_{\mathbb{Q}}\mathbb{Q}\,(\pi, K)\,.$$

4. Recall from the cornucopia of Izrail Gradshteyn and Iosif Ryzhik [[GR94], formula 8.129 (MO130) p.913], the value of K at this normal modulus given by

$$K = K(\frac{\sqrt{2}}{2}) = \frac{\Gamma\,(\frac{1}{4})^2}{4\sqrt{\pi}}.$$

5. Invoke the algebraic independence of π and $\Gamma(\frac{1}{4})$ proof by the Chudnovski around 1977, recently improved by Yuri Nesterenko ([NP96] p48), to conclude from the previous point.

The reader should easily build-up the desired proof of the algebraic independence of π and $E(\frac{\sqrt{2}}{2})$ by applying the merging or gluing principle inferently to the arguments of those steps.

♦

More relevantly, the author invites the reader to meditate on a geometrico-motivic interpretation of the Legendre formula and also on the complexity drop of the concerned periods reflecting the symetric nature of the normal modulus, given by the formula $k = k'$.

Remark. Using the formula $K(\frac{2\sqrt{k}}{1+k}) = (1+k)K(k)$, we get as a free bonus, a countable number of algebraic independencies of the complete elliptic integral of the first kind with π, namely

$$trdeg_{\mathbb{Q}}\mathbb{Q}\left(\pi, K(k_n)\right) = 2$$

with $(k_n)_{n\in\mathbb{N}}$ defined by $k_0 = \frac{\sqrt{2}}{2}$ and $k_{n+1} = \frac{2\sqrt{k_n}}{1+k_n}$.

Note also that those algebraic independences may be obtained directly from the above cited source ([NP96] p48), where a measure of algebraic independence of periods of CM elliptic curves is given.

2.2.6 Varieties in the 3−dimensional euclidean space : algebraic surfaces.

We will move-up one dimension and consider some cases of surfaces, as we mentioned earlier in the introduction, those surfaces constitute an intermediate complexity class of varieties : an important proportion of recent research efforts concentrate on some subclasses of this family of algebraic varieties like the one of $K3$ surfaces.

Ellipsoids.

Analogously to the case of the family of ellipses $(\mathcal{E}_{a,b})$, we consider their 3-dimensional deployments; through the family of the centered ellipsoids parametrized by triples (a, b, c) of three strictly positive real numbers a, b and c, with $a > b > c$. A classic computation gives for the

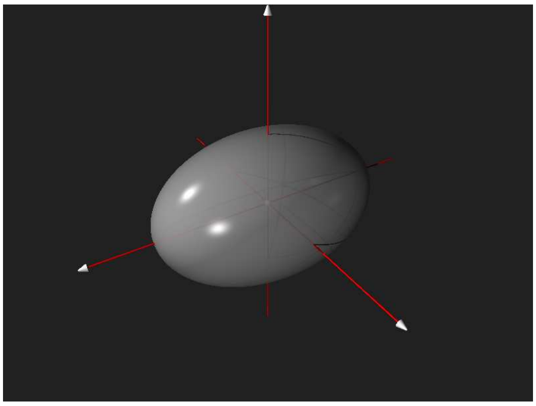


Figure 2.10: Ellipsoids $(\mathcal{E}_{a,\,b,\,c}) : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$

first period the following expression.

$$\mu_{a,b,c} = \mu(\mathcal{E}_{a,\,b,\,c}) = \frac{4}{3}\,\pi\, abc.$$

For the second period; that is, the surface area of the ellipsoid, the computation is again more tedious; after several unsuccessful attempts, it is by rambling in a mathematic library that the author came across the book of Derek Lawden [[Law89], p.102], in which the result was found computed : it involves the incomplete elliptic integrals of the first two kinds **F** and **E** and writes as follows; the parameters and arguments of these functions being explained inside the two bracket delimiters that follow the equation; note, in this regard, that the parameter k verifies $|k| < 1$.

$$\lambda_{a,b,c} = \lambda(\mathcal{E}_{a,b,c}) = 2\pi c^2 + \frac{2\pi\, ab}{\sin\phi}[\cos^2\phi\mathbf{F}(\phi,k) + \sin^2\phi\mathbf{E}(\phi,k)]. \quad \left(\cos\phi = \frac{c}{a}. \quad k^2 = \frac{a^2(b^2-c^2)}{b^2(a^2-c^2)}.\right)$$

Recall that the elliptic integrals **F** and **E** are given by the following expressions.

$$\mathbf{F}(\phi,k) = \int_0^\phi \frac{1}{\sqrt{1-k^2\sin^2 t}}\,dt \quad \text{and} \quad \mathbf{E}(\phi,k) = \int_0^\phi \sqrt{1-k^2\sin^2 t}\,dt \quad \text{with} \quad |k| < 1.$$

The mean point of the ellipsoid is the origin, the transcendental degree of the two fundamental arithmetic periods is given by the following formula.

$$trdeg_{\mathbb{Q}}\mathbb{Q}\left(\mu_{a,b,c},\ \lambda_{a,b,c}\right) = trdeg_{\mathbb{Q}}\mathbb{Q}\left(\pi abc,\ \pi c^2 + \frac{\pi\, ab}{\sin\phi}[\cos^2\phi\mathbf{F}(\phi,k) + \sin^2\phi\mathbf{E}(\phi,k)]\right).$$

Suppose this transcendental degree maximal then both periods are transcendental, for the first one we get the fact that $abc \notin \frac{1}{\pi}\bar{\mathbb{Q}}$, as we obtained for the case of ellipses; and similarly, the second intricate period is unexploitable in this bad looking form, lacking symmetry.

Let us move on anyway looking for other simplifications : if we take the three parameters a, b and c rational then this transcendental degree simplifies to the following expression.

$$trdeg_{\mathbb{Q}}\mathbb{Q}\left(\mu_{a,b,c},\ \lambda_{a,b,c}\right) = trdeg_{\mathbb{Q}}\mathbb{Q}\left(\pi,\ \cos^2\phi\,\mathbf{F}(\phi,k) + \sin^2\phi\,\mathbf{E}(\phi,k)\right).$$

When the parameters (a,b,c) are rational, the surface $(\mathcal{E}_{a,b,c})$ admits non-trivial rational points. The stable criterion imposes, therefore, the algebraic dependence of $\mu(\mathcal{E}_{a,\,b,\,c})$ and $\lambda(\mathcal{E}_{a,\,b,\,c})$ which is equivalent after multiplicative simplification modulo $\overline{\mathbb{Q}}^*$ to that of π and $\cos^2\phi\,\mathbf{F}(\phi,k) + \sin^2\phi\,\mathbf{E}(\phi,k)$.

If we consider the even more trivialized or degenerate cases of two equal semi-axis $b = c$, then $k = 0$ and the complexity of the fundamental arithmetic periods tuple drops down even more, reflecting again as the preceding case of ellipses, the symmetry increase of the surface, so that the previous algebraic dependence is equivalent to that of π and $\phi = \arccos\frac{c}{a}$, in more precise transcendental terms we have :

$$trdeg_{\mathbb{Q}}\mathbb{Q}\left(\mu_{a,b,c},\ \lambda_{a,b,c}\right) = trdeg_{\mathbb{Q}}\mathbb{Q}\left(\pi, \arccos\tfrac{c}{a}\right).$$

Since that for all real x of $[-1,1]$, $\arccos(x) = -\imath\ln(x+\sqrt{x^2-1})$, we are lead to the study of the algebraic independence of logarithms of algebraic numbers; as we have $\ln -1 = \imath\pi$. Actually in Transcendental Number Theory, it is a famous conjecture that the logarithms of $\mathbb{Q}-$linearly independent algebraic numbers are expected to be $\mathbb{Q}-$algebraically independent.

Moreover, similarly to the case of the family of ellipses $(\mathcal{E}_{a,b})$ with rational semi-axis a and b, the same countability argument conforts that those periods numbers are likely to be algebraically independent and we find out again that two periods may be not enough to get a robust arithmetical discriminant; in this particular case : the on-going elaboration of extended versions of the criteria of this memoir fix that; mainly by considering more than two periods, as already mentioned.

Euler-Fermat surfaces.

We study a family of bi-quadratic or quartic surfaces, that are deployments out to the 3-dimensional euclidean space of the Fermat curve $\mathcal{F}_4$ of degree 4, and parametrized by a real number a which one can suppose, without loss of generality, strictly positive; bearing in mind their equation appearing under the graphical representation below. This equation was studied by Euler but it was also considered by Fermat; hence the reconciling qualifier « Euler-Fermat »; at that time,large parts of mathematics were more considered as entertaining puzzles games; and mathematicians across Europe would challenge each others with open problems, either privately in personal letters or in public during academic or official sessions.

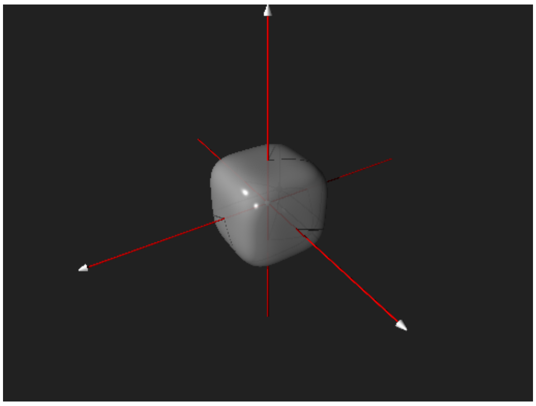


Figure 2.11: Euler Fermat quartics $(\mathcal{Q}_a) : x^4 + y^4 + z^4 = a^4$.

The first period was computed starting from tables of the gathering collective russian summa of Anatoly Prudnikov, Yuri Brychkov and Oleg Marichev [[PBM86], p.383], indeed, this integral turns out to be a special case of Dirichlet integrals and a rather quick computation gives

$$\mu(a) = \mu(\mathcal{Q}_a) = \frac{a^3}{6\sqrt{2}} \frac{\Gamma^4\left(\frac{1}{4}\right)}{\pi}.$$

The second one is, as usual; not as straightforward; no close integral expression was found although there should be a rather tractable way of getting one. All those second "reluctant" periods considered in this memoir may actually be particular cases of the studied ones by Emre Sertoz in his recent 2020 thesis; from Max Planck Institute of Bonn; whose title comprises the terms "Periods of hypersurfaces", and where he shows how Ordinary Differential Equations (ODE) monodromy techniques can be wisely used to compute periods of varieties starting from moduli of higher dimensional varieties that encompass them. The curious reader might find this cutting-edge computational text by searching the Internet for its author name; and found out in it, how its author has compared some expensive propriatory CAS (Computer Algebra Systems) to the free Open Source one; Sage; the latter Open Source alternative being the overall winner in terms of ease of access and performances; generally Open Source Free Software is the best choice when it comes to Academic and Research computations.

Here, the result was not expressed in the simplest way; despite our efforts, we just managed to get, after applying the Gauß Ostrogradsky formula also called the divergence formula, the expression below.

$$\lambda(a) = \lambda(\mathcal{Q}_a) = 8 \iiint_{[0,a]^3} \frac{x^3 + y^3 + z^3}{\sqrt{x^9 + y^9 + z^9}}\, dx\, dy\, dz.$$

Bearing in mind that this surface has for obvious mean point the origin and that for a rational, it admits rational points, that seem to be only trivial ones; the stable criterion would fit this case if the two periods were algebraically independent, namely $\frac{\Gamma\left(\frac{1}{4}\right)}{\pi}$ and $\iiint_{[0,a]^3} \frac{x^3+y^3+z^3}{\sqrt{x^9+y^9+z^9}}\, dx\, dy\, dz$.

To the best of our knowledge, the first number is transcendental, because of the algebraic independence established by the Chudnovski and recently improved by Yuri Nesterenko, of $\Gamma\left(\frac{1}{4}\right)$ and π. For further developments on this subject, see the first articles of the fourth Number Theory booklet of the russian EMS or the Encyclopœdia of Mathematics Sciences [NP96].

2.3 Perspectives.

We give in this section that may appeal to Number Theory researchers, some ideas of exploratory paths, concerning the stable criterion, it is a list that opens the field of future investigations.

2.3.1 Theoretical consistency of the criteria.

In the context of a clustered and enclosed hypersurface V, the required birational invariance of the stable criterion would be conforted by the algebraic invariance of the integrals of its volume forms. If a birational image of V, noted $f(V)$ satisfies the same hypotheses as V, it is desirable that the resulting image periods come out being algebraically dependent to the initial periods of V, either by being given by an algebraic function of the two initial periods or by being effectively algebraically dependent to those initial ones; which can be formalized by the equality

$$\mathcal{P}_{\mathcal{A}}(f(V)) = g_f(\mathcal{P}_{\mathcal{A}}(V)). \tag{2.5}$$

Here g_f is an algebraic expression depending on the birational map f. The wished invariance behavior of the fundamental arithmetic periods under the action of birational maps is more precisely formulated in terms of transcendental degrees as follows :

$$trdeg_{\mathbb{Q}}\mathbb{Q}(\mathcal{P}_{\mathcal{A}}(f(V))) = trdeg_{\mathbb{Q}}\mathbb{Q}(\mathcal{P}_{\mathcal{A}}(V)). \tag{2.6}$$

Remark. The formula(2.1) page 23 giving the mean point requires the computation of n periods, n being the dimension of the ambient Euclidean space. The algebraic assumption of this point is therefore quite restrictive. The author is actually exploring this issue.

Once the algebraicity of the mean point is ensured, we then focus on the two periods λ and μ. For the previous consistency, the coordinates of $G_{f(V)}$ should also be related algebraically to those of G_V, ideally we should have a formula like

$$G_{f(V)} = h_f(G_V). \tag{2.7}$$

Here again, h_f should be an algebraic expression depending on the birational transformation f. In an ideal mathematical world; we would simply have $h_f = f$ and then the following equality.

$$G_{f(V)} = f(G_V). \tag{2.8}$$

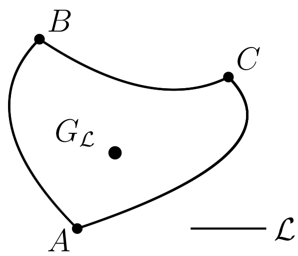


Figure 2.12: An enclosed and piecewise smooth plane curve with its mean point.

Unfortunately, this ideal equality (2.8) turns out to be very unlikely satisfied, the preceding one (2.7) is either unlikely satisfied, even the previous transcendental degree one (2.6) about fundamental arithmetical periods might not be fulfilled. So what is left ? After some recent sporadic meditations about that birational invariance issue, what might be preserved under the action of birational maps is the transcendental degree of the whole set of generalized periods considered : the mean-point ones plus the fundamental arithmetic periods. Separating the set into two subsets kicks away the birational invariance, keeping the whole set preserves it; as it was the case for the arithmetic criteria considered here, the more the periods the better and finer is the capturing of arithmetic properties of the studied varieties; in other words, periods are the lenses for scrutining the Arithmetic of varieties, the more the periods, the better is the resolution of the Arithmetic microscope.

All this leads to a reformulation of those criteria : either including the mean-point in the transcendental maximality requirement or replacing the transcendental degree maximality requirement by an inequality might be another way to go; actually, the author has examined both paths and is rather confident about their way-out, the first one being the safe bet.

To conclude this consistency paragraph, let us notice that some elementary transformations of the euclidean ambiant space force to patch again the considered periods with others ones. For the elementary case of a simple loop $(\mathcal{L})$ of the euclidean plane; the only two fundamental arithmetic ones $\lambda(\mathcal{L})$ and $\mu(\mathcal{L})$ are obviously invariant under mere rotations about the mean point $C_G(\mathcal{L})$; while the presence of rational points is clearly not invariant under such rotations but if we stick to algebraic curves defined over $\mathbb{Q}$, this rotation argument does not hold anymore.

2.3.2 Perspectives for forecoming approaches.

In the algebraic case.

• Investigate the Theory of Diophantine Approximations with the theorems of Roth and Dyson; having in mind, the simultaneous approximation of the two periods λ and μ by algebraic numbers; together with a possible corresponding parallel process of approximating general varieties by algebraic ones.

• On the Hodge and De-Rham side; with Hodge cycles on varieties; both on the studied varieties and the corresponding periods variety; that is the variety or space, built from periods of the initial varieties through a promising periods functor; this periods approach should go along the paths developed in articles of Goro Shimura [Shi80] on the algebraic independence of periods of abelian varieties; and in the footsteps of results of Pierre Deligne [Del80] about Hodge theory and the periods of abelian varieties; as well as along the corpus avenues of periods domains traced by Philip Griffiths in his periods domains monographs.

A fairly exhaustive exposition of the periods domains theory may be found in another collective work "summa", namely the book "Periods domains and periods mappings" [CMSP17] by James Carlson, Stephan Müller-Stach; and Chris Peters, published by Cambridge University Press. This might be wisely completed with a recent key article of Annette Huber and Gisbert Wüstholz on periods of 1−motives.

• On the motivic side; the most exciting approach; but also the most conjectural. Considering $K = k(\mathcal{P}(V))$, with $\mathcal{P}(V)$ denoting a meticulously chosen set of generalized periods of V to be specified; then studying V_K and seeing if $Br(V_K)$ can be related to $Ш(J(V_K))$, the Tate Shafarevich group of $J(V_K)$, the Jacobian of V_K; through $G_{mot}(V_K)$, the promising motivic Galois group of V_K.

Seeing then, where $Br(K)$ can play a role, perhaps by quotienting either by $Br(V_K)$ or $Br(K)$ the other concerned groups in those lines, seeing also; how $Br(K) = Br(k(\mathcal{P}(V)))$ that may be renamed "the modular Brauer group of the variety V" or "Brauer group of periods of the variety V" can be expressed in terms of the various invariants of the variety V, ideally the motivic ones. The key bridging Grothendieck notions between Brauer groups and motivic Galois groups would be the tensor product, the notion of torsors and finally cohomology.

What can be roughly said, according to the definitions, is that the bigger is the Brauer group, the fewer are the rational points. And following our conjecture; the presence of rational points on a variety belonging to a family of varieties should be detected by the size drop of its motivic Galois group relatively to the average one or general case; since it is expected that we have a result close to the following equality; where the "size" should be either related to a basic dimensional integer or related to some group theory index number.

$$trdeg_{\mathbb{Q}} K = size_{\mathbb{Q}} G_{mot}(V_K).$$

This size drop should also be proportional to the density of rational points on the corresponding variety, and the qualitative aspect of the size drop should reflect the nature of those rational points; for instance a steeper decrease going along with the presence of trivial rational points; or with integers points and finally with the presence of whole rational or integers subvarieties, for the steepest drops cases; the term integers refering to the ring $\mathbb{Z}$.

Here the vaguely coined "arithmetic shape" in the beginning of this memoir, may be renamed as the $\mathbb{Q}-$shape of the studied family of varieties and should be revealed or "measured" by the properties of the promising motivic Galois groups over $\mathbb{Q}$ of those varieties; possibly by a kind of "higher level" or "higher topology or geometry", defined on the category of those motivic Galois groups of varieties.

The author suggests to rename those into Galois groups of varieties or Galois-Grothendieck groups of varieties, dropping the "motivic" adjective for mathematicians names; groups of an hypothetical Galois Grothendieck Group Theory and the shortcut TGT would better fit some subject classification abbreviation or informal communications between scholars, researchers, experts and PHD students.

Why suggesting the term Galois group of varieties for the motivic Galois group ? Well, it is a way of restoring Galois legacy through an extension of classical Galois theory to algebraic varieties. Classical Galois theory answered the question of solvability by radicals of a one variable polynomial equation; the motivic Galois group does the same through the proposed criteria for the question of solvability in rational numbers of several-variables polynomial equations, providing the ultimate obstruction to it. The motivic Galois group may be thought as a multidimensional version of the classical Galois group.

A relevant recent reference for all that trend of ideas is, once again; the already cited, freshly baked book of one of the numerous german arithmetic geometry schools; namely the bavarian or "bayerish" arithmetic geometry school book on periods of Nori motives [HFvWMS18]; by Annette Huber; Stephan Müller Stach, Benjamin Friedrich and Jonas von Wangenheim : this key monograph will probably constitute the future reference on the periods aspect of motives. Again, the curious reader may have a look at preliminary versions of that relevant "summa" by searching the Internet for the first author name.

In the general case : going towards analytic varieties, convex bodies and compact connected spaces.

• Explore the link between the two parts of this memoir with the connected components. In this trend, it would be the topology of a variety that has a greater impact on its arithmetic rather than its differential geometry as often loudly claimed in Arithmetic Geometry books. See the validity of other hypotheses for the criteria.

• Replace clustered and enclosed by compact, connected and without boundary; that is going towards Milnor theory and Riemanian geometry frameworks; focusing on Euler Poincaré characteristic techniques; and figuring out an arithmetic equivalent of that fundamental topological invariant; maybe built from the transcendental degrees of the fields generated by suitable subsets of fundamental arithmetic periods that would be called special fundamental arithmetic periods; a piece of the hypothetical periods space related to a conjectural extension of the motivic Galois group to analytic varieties.

Conclusion

The fate of the criteria, as exposed here, despite some corrections added to the english version of the initial clumsy french version, is uncertain; when applied to ellipsoids, the version of the stable criterion given here, rubbs a bit harshly against the algebraic independence conjecture of logarithms of algebraic numbers—see the article of Michel Waldschmidt [[Wal06], p.28] —as well as a conjecture of algebraic independence of periods and quasiperiods of non-CM elliptic curves (ie curves without complex multiplication); see Yves André's book [[And04], p.84]—when applied to ellipses; but there is good hope that the on going smoothening and polishing of their formulations can strengthen them through extended versions to make those fit these conjectures and even generalize them; as well as provide eventually inspiring ideas within a totally a new framework coined as Hypergeometry for the study of rational points; possibly leaving a few pioneering ideas to future generations of curious number theorists surfing the world wide web, coming across this cryptic memoir hidden in an electronic bottle, thrown away into the vast oceans of the Internet.

Chapter 3

Appendix

3.1 Lifts for Transcendental degree.

3.1.1 Functional lift.

The criteria considered in this memoire rely on the determination of the transcendental degree of a tuple of hypergeometric functions at algebraic arguments [conclusion of (2.4) p42], leading to the extension of the same question to a tuple of general transcendental functions values.

Given a tuple $(f_1, ..., f_n)$ of such functions, we want to evaluate its transcendental degree at the special or ponctual algebraic argument tuple $(\alpha_1, ..., \alpha_n)$, that is

$$trdeg_{\mathbb{Q}}\mathbb{Q}(f_i(\alpha_i)). \tag{3.1}$$

The first way to do that is to lift the ponctual problem to the functional setting by evaluating

$$trdeg_{\mathbb{Q}(z)}\mathbb{Q}(z)(f_i).$$

Indeed, shifting to functions spaces is a dynamisation process that usually unseizes the eventually stuck ponctual case. The key requirement of this method is that its needs a prior uniformizing reduction of the arguments to the diagonal $\alpha_i = \alpha$, leading to $(\alpha, ..., \alpha)$ and therefore to restricting the evaluation of (3.1) to

$$trdeg_{\mathbb{Q}}\mathbb{Q}(f_i(\alpha)).$$

When this key prior uniformizing step, fitting the general uniformization trend of this memoir; is achieved, we always have at least an evaluation of the sought transcendental degree through an upper bound

$$trdeg_{\mathbb{Q}}\mathbb{Q}(f_i(\alpha)) \leq trdeg_{\mathbb{Q}(z)}\mathbb{Q}(z)(f_i).$$

For our criteria a lower bound would be more relevant.

Now remains to tackle the uniformization of arguments; one quick way to get it is to make a simple diagonal change of variables, by rescaling them with the appropriate algebraic numbers; provided that the considered class of functions is stable under those algebraic rescaling.

3.1.2 Uniformization principle.

An illustration of the uniformization trend of thoughts of this memoire that is meaningfull for our criteria would be the conjecture of the algebraic independence of π and odd zeta values. Recall that this conjecture with the criteria proposed in this memoire would give Fermat theorem.

Conjecture 4. *Let $N \geq 1$ be an integer then π and the first $N-th$ odd zeta values are algebraically independent :*

$$trdeg_{\mathbb{Q}}\mathbb{Q}(\pi, \zeta(3), \ldots, \zeta(2N+1)) = N + 1.$$

How this notorious expected-result can be obtained ? A possible approach would consist in a four stages process.

First step : notice that π can be replaced by its square π^2 and hence by $\zeta(2)$ using Euler Basel theorem $6\zeta(2) = \pi^2$.

Second step: notice that for all integer $n \geq 2$, we have $\zeta(n) = Li_n(1)$ giving a uniformization of arguments through polylogarithms.

Third step : the hypergeometric uniformization of polylogarithms with a big depth giving

$$Li_n(1) = {}_{n+1}F_n(\{1\}^{n+1}; \{2\}^n | 1)$$

where the $k-$times repeated parameter $a, ..., a$ is often denoted by $\{a\}^k$ in the litterature.

The main advantage of the hypergeometric expression is that an algebraic relation involving single variable hypergeometric functions is likely linearizable into a linear dependence involving only multivariable hypergeometric functions, in other words; it is likely linearizable within the same intrinsic class of functions. We call this process (MLP) for Multivariable Linearization Principle. Moreover if the values are taken at the unit argument, like for zeta values, we might be able to remain in the single variable setting. Notice by the way, this MLP principle would coroborate the fact that polynomial expressions in single zeta values are linearizable into linear expressions in multizetas values. In the same way, the transcendental degree of periods of algebraic varieties defined over $\mathbb{Q}$ would therefore be reduced to the rank of a linear vector space of "multihypergeometric" values, providing the already mentionned linearization of the Grothendieck Period Conjecture or (GPC) .

Fourth step : now two options fork; one ODE based on Differential Galois theory and one involving motives. Precisely,

a) either an hypergeometric or polylogarithm version of Siegel-Shidlovsky theorem with a lower bound estimate of transcendental degrees, would help to conclude but it would be a tough process to get it; indeed, the available Shidlovsky theorem applies only to specific hypergeometric functions of ${}_p\mathcal{H}_q(\mathbb{Q})$; precisely among the confluent ones, meaning $p \leq q$, with non resonant parameters requiring in particular that the $a_i - b_j$ are not integers for the algebraic independence conclusive known cases.

Nonetheless, the wealth of relations within the hypergeometric functions spaces is so unlimited that it may be possible to circumvent those requirements by suitable transformations of both parameters and arguments. It is there that *the strong uniformization* comes into place : we might be able to go down to small "lengths" $p+q$, expressing periods of algebraic varieties defined over $\mathbb{Q}$ as values of algebraic functions at hypergeometric values of small lenghts. So those small lenghts hypergeometric values may be thought as building blocks of those periods; and this phenomenom should have a motivic interpretation. The main point to retain is that those small lenghts hypergeometric values may open the Siegel-Sidlovksky functional gate to an alternate route towards the linerarization of the (GPC), through ranks of sheaves on hypergeometric D-modules; the other route being through ranks of $\mathbb{Q}$-linear vector spaces generated by "multihypergeometric" values.

b) or a more likely efficient approach would take the path of motives.

3.1.3 Motivic lift.

Another related method for evaluating the transcendental degree of special values of transcendental functions would be their motivic lift. The motivic lift is a method that fructified from the recent maturation of the theory of Motives and as such, may be more prone to be a successful approach, indeed, the very rich algebraic structures of motivic objects make it quite powerful for extracting relations between those, and then going down to the initial non-motivic objects.

It firstly came in the form of the lift of periods to motivic periods, then it was extended to the lift of special values to motivic values, as soon as the initial studied values could be interpreted as periods, this prior interpretation step being a critical one.

The main drawback of the motivic lift for algebraic independence and transcendental degree issues is that it is conditional to the Grothendieck Periods Conjecture (GPC); but for simpler *linear* dependence issues, the motivic lift may actually be successful : Francis Brown obtained his *linear dependence* (2,3)-theorem, by lifting multiple-zeta values (MZVs) to motivic MZVs; the conjectural *linear independence* counterpart of this result, giving the (2,3)-*linear basis* positive answer to the conjecture of Hoffman-Zagier about the $\mathbb{Q}$-linear structure generated by those MZVs and therefore the sought algebraic independence of odd zeta values after applying the MLP principle. A motivic-cohomology obstruction to the linear dependence of the (2,3)-generating family might provide its independence and therefore, turn it into a basis.

For our concern about the proposed arithmetic criteria and the peculiar case of the odd zeta values algebraic independence, we would suggest a motivic lift of hypergeometric values to motivic hypergeometric values, more likely successful after the multivariable *linearization* process (MLP) than before; as evoked for the case of MZVs; but both paths may be worthy to explore. As noticed, this needs the prior condition that hypergeometric values can be expressed as periods; while the converse was already claimed as the main uniformization principle of this memoire. Fortunately, this prior condition is easily seen to be fulfilled through the Euler integral representation of hypergeometric functions.

Finally, for the linear relations issues of periods coming from 1-motives, the theory is well geometrically settled as exposed in the paper of Annette Huber and Gisbert Wustholz on 1-motives : linear relations among periods of 1-motives come from geometry, confirming the GPC.

Three recent papers of young researchers explore those motivic lifts considered here.

1) Clément Dupont with an appendix from Don Zagier. In this paper the first author lifted odd zeta values to motivic ones, to tackle the study of ... *linear* forms in those odd zeta values. arXiv.org:1601.00950

2) Francis Brown and Clément Dupont for Appell-Lauricella hypergeometric functions.

arXiv.org:1907.06603v2

3) Clément Dupont and Javier Fresan for Polylogarithms. arXiv.org:2305.00789v1

3.1.4 The conceptual link between the two lifts : structuring relations.

The slogan is : algebraic relations within both hypergeometric functions and their values at rational or algebraic numbers; have a geometric origin.

This would give a structural or motivic meaning for the billions of hypergeometric formulas crawling in thousands of books and billions of articles on special functions. Moreover, if we take for granted the main uniformization principle of this memoire that algebraic varieties over $\mathbb{Q}$ have periods given by algebraic expressions in hypergeometric values; this would coroborate the Grothendieck Periods Conjecture philosophy that algebraic relations among classical or effective periods come from geometry.

Regarding relations between functions, they are of two types :

1) The horizontal ones between functions of the same class; and

2) The vertical ones, between functions belonging to intrinsically different classes.

A way to structure the whole package is to find the smallest structured class containing all the targeted classes.

Example : Single variable Hypergeometric functions with algebraic parameters satisfying specific resonance conditions, are peculiar G-functions, and the latter are peculiar arithmetic Gevrey series.

The main structuring tools ("les tamis classifiant" in french) come from the theory of Differential Equations and Operators through D-modules and their sheaves; the motivic and arithmetic nature of the classification goes along expansions into series over $\bar{\mathbb{Q}}$ of the solutions-sheaves; mostly from bounding conditions on growths-sizes of those expansions, in particular Galoisian heights growths for specific classes like E and G-functions; giving the rigidity that allows punctual-to-global statements such as the Siegel-Shidlovski theorem.

One way to unify the functional approach to the geometric one is through the D-module theory of connexions and especially Gauss-Manin connexions in the case of periods investigations.

3.1.5 Hypergeometry.

Hypergeometric functions behave relatively well under basic ring operations on functions together with integration and derivation, but they are likely unstable under composition and multiplicative inverse as classical periods, this is the source of the quasi-stability of the periods functor on hypergeometric varieties.

A hasty conclusion of this memoir would be the following linearizations of the (GPC) and the proposed arithmetic criteria for rational points on algebraic varieties.

The absence of non-trivial rational points on an algebraic variety V defined over $\mathbb{Q}$ is governed by the algebraic independence of hypergeometric values coming from periods of V or equivalently by the corresponding linear independence of multihypergeometric values.

3.2 Biographical and historical notes on the criterion genesis.

1. Around December 1992. Research about Fermat conjecture; as a fresh 21 years naive undergraduate; while rambling "in the wild" landing in a public library ("La Vilette : Cité des Sciences", Paris), I came across the Modularity conjecture and its link to Fermat conjecture, resulting from works of Gerhard Frey, Yves Hellegouarch and others; in a review of a paper of Barry Mazur and Kenneth Ribet from a sample of the French Mathematical Society (SMF) research monograph Astérisque. In early 1993, going back to University for administrative issues (absence in lectures courses sessions); I urged, about six months before the public release of Fermat's conjecture proof; some faculty (Professors Jean François Mestre and others) to look for a proof of Fermat conjecture; since it was clear for me that the proof was within reach.

2. Around 1993 : First stumbling steps in research; tackling research level open problems with original and personal ideas; while writing "autonomously" mathematics since about 1990, naively trying to prove by myself all the mathematical results I came across.

3. Around 1996. After some administrative issues with the head of the mathematics department, perceiving a bad oriented talk escalation; I left the academic world for the study of motorcycles petrol engines and mechanics; while keeping contact with up-to-date research. I passed the motorcycle driving license and bought a second-hand Suzuki off-road long-range travel vessel, namely the well suited "big DR" or "DR Big" or "Suzuki DR750 Desert Express" with deserts crossing travels in mind. I actually crossed a desert; but it was not a stony and sandy one...

4. Around 2000. Coming-back from two years of hardships in hospitals, rehabilitation centers and medical institutes ; to the less hazardous University for the Agregation after a quite severe and almost deadly motorcycle crash that ended up with a total then, fortunately a partial quadriplegia and a completely smashed body; broken from head to toe. Relearned handwriting then keyboard typing to finally handle LaTeX and study GNU/Linux systems; while investigating weak version of the criterion in the δ form. Obviously, I roughly failed the Agregation (the D−day, there was a strike of public transportation, so I had to wake up at about 4h am to hope to get to the faraway exam center; when I finally arrived; I was so exhausted that I felt asleep during most of it. . . I learned later that all the hotels around were full and booked quite early in the year by wisely prepared students; most of them, being from the academic "serail"; what a gap with my "maverick" situation), in fact; all

that year I was wondering why I took this curriculum; the mathematical deep thinking was parsecs away from the technical exigences of that "Concours", my crippled hands did not neither fit the scope of speed and "virtuosity" required for that quite demanding national competition.

5. Around 2003. Counter-examples to the weak version of the criterion. Case of genus 0 curves with basic plane conics : circles and ellipses. Back to University for a Master 2 in mathematics.

6. Around 2004. First improvement of weak criterion : algebraic independence of periods.

7. Problem of the center of the "enclosed" variety raised by Michel Waldschmidt during a quick informal discussion in "Chevaleret Mathematics Institute" couloirs.

8. Second improvement by requiring an algebraic center for the "enclosed" variety.

9. Around 2005. Vain search rambling again "in the wild" of a public library (BPI Beaubourg du Centre George Pompidou, Paris) for an equivalent of the criterion in the complete collected works of Jean Delsarte, Jacques Hadamard, Charles Hermite, Camille Jordan, Gaston Julia, Joseph-Louis Lagrange, Pierre-Simon De Laplace, Adrien-Marie Legendre, Émile Picard, Henri Poincaré and André Weil.

10. Counter-example to the generalized version as stated here. Begun the study of arabic with modern translated versions of the Qu'ran.

11. Around 2007. Development and study of a robust version of the stable criterion. Starting to settle a modern formulation of it, by digging into the bulk of algebraic geometry theoretical abstract-nonsense.

12. Around 2009. While pursuing the modern formulation of the criterion, wrote some system programs for the GNU/Linux based platforms. During that period, in early summer 2009, a digital disaster stroke four computers carrying this activity : all their hard drives containing about two years of mathematics investigations and Free Software code crashed simultaneously with the same symptoms. Left research and mathematics for the study of computers hardware and digital security threats. Back to "two wheelers" tech : riding bicycles for the first time again before an average mileage of 35000 kms or 20000 miles in about two years from early 2009 to fall 2011; kissing seriously the ground a few dozen times; thrice quite severely, ending up again in emergency units to be checked (head trauma) and resewed all over injuries locations and cracked bones.

13. Around 2011 : exhausted by bicycle tribulations; some reaching deep forests with quite steep hill climbs in 100 miles/day intensive rides; back again to motorcycles and mechanics issues while maintaining a second hand purchased motorcycle, another Suzuki one : the basic Suzuki GS500. Made the initially intended journey of 5000 miles to the semi-desert region of Morocco of my ancestors. Creating various related blogs while another quite serious motorcycle crash stopped any activity for a while in early 2012. Fixed the crashed Suzuki GS 500 motorcycle to ride it again a few months later.

14. Around 2016. Starting a motorcycles restoration workshop project : learning engineering and craft techniques; gathering tools sets, looking for workshops. Conceived and tested a small chemical kit of "pre-catalyse", that can be easily plugged into the air-box of petrol engines to reduce hydrocarbons emissions during cold starts while increasing engine performances. Made several intensive highway journeys across France for the kit tests and looking for workshops.

15. Around 2020. In the context of 2020 sanitary curfews, paused the motorcycles project, and after an online registration to give mathematics courses forcing me back to mathematics, I relearned the basics, starting from linear algebra courses on a smartphone and then soon diving into the research universe after the "math-mode" took-off; begun excavating about 10000 pages of past research paper notes out of a first stack among others.

    Considering salvaging the stuck hard-drives to put all their left contents both mathematics and Free Software on-line.

    Translation into english of the initial french memoir in this document; considering also translating and putting on-line all that what was left just before the digital blackout of 2009; as well as new developments of the criterion by either fitting it into the motivic context, or more likely by introducing a totally new framework with a good hope of helping and/or maybe leaving a few inspiring ideas to future generations of arithmeticians.

3.3 Epilogue : dedicace, aknowledgements and final words.

Dedicace

This translation is dedicated to those who may have felt more a less neglected while I was busy; being deeply absorbed in quite heteroclite activities; firstly to the Essential bounty, namely my Parents then to my near family, my sisters and my brother and to the far away family in Morocco and elsewhere; and then, last but not least; to all the anonymous folks all around the world that made significant contributions in developing human knowledge, either by enriching its contents and/or tools, or by spreading it or maintaining its physical (working-staff) or digital (computers-staff) places, without ever being quoted.

Aknowledgments

Obviously, this memoir owes nearly everything to the mathematicians community past and present, especially to mathematicians that bother themselves to put online courses notes, slides and surveys; with the results of their research work, mostly in the form of preliminary versions of articles, papers, books and/or monographs. Some official platforms like ArXiv and other more controversial digitalization ones have been crucial during the recompilation process of past research notes.

Technically, this translation was again elaborated mostly on GNU/Linux systems; mainly with the GNU/Linux distributions Debian and Mint; the first being the trusty long term assistant; oddly, it always ended up being the most efficient in terms of workflow. I reiterate the handshake of the beginning to the Free Software communities of GNU/Linux, Debian and Mint; including *TeX* and *LaTeX* communities.

I also used Android systems on smartphones and tvboxes to produce pieces of this memoir, but I tweaked them back to their GNU/Linux origins; so also a handshake to the Free Android apps builders. Termux, Xplore and Q-edit were the most used apps for editing and compiling *LaTeX* files. Some graphs were obtained in both environments with Sage, GeoGebra, Maxima; together with Genius Maths Tool for GNU/Linux systems; and finally Grapher Free, Maths Grapher 3D for Android systems. So an equal handshake to the developers communities of all those softwares.

Most of those years; I was partially and sporadically supported by the county I live in, the "Département des Yvelines" since the age of 9; when I received a cash prize with a bank account, from them "for school excellence", after wining a math oral quiz contest; actually, I was signaled to the administration at the age of 3, when the kinder garden teacher was quite surprised in front of uncommon handling of letters, words and numbers as well as 3D-drawings of basic structures like houses and cars.

Final words

About mathematics in general : the purpose of tackling directly very concrete elementary examples; in this memoir, is to advocate and promote this way of doing mathematics; that seems to get back to the front scene, with the new generations of mathematicians born during the "digital or computers era" : the author is personally convinced that computer sciences will take over and even produce pure mathematics; but in a quite, quite far away future; with the symbiotic long term development of quantum computers mainframes and artificial intelligence (q-bits meeting multi-valued truth of some logic topos); and that mathematicians should, for now, care more about giving-right back numerous useful results to their hosting societies, like applications, motivational and/or explicit elementary examples of the theories they elaborate; in order to perpetuate their activity that is under the long term threat of machines : modern societies are in general not shy in cutting research budget reserved to highly theoretical or abstract research fields; foremost if they do not get "their cash back" in terms of fruitful applications. Those suggestive points, should actually be seen as giving possible inspirational reviving kicks to the seemingly slightly stuck pure part of the antediluvian venerable Number Theory that is nowadays losing attractiveness at a worrying pace, in front of its vivid and dynamic modern digital fields of applications : the prehistorical way of outputting mathematics still dominating nowadays; with a terrific lack of synergy despite the digital era ease of instant communications should be reconsidered under the hospices of computer sciences structures.

Meditating about those speculations and the previous narrated piece of life trajectory @ the blinking shell prompt of the well named Bash (the Bourne Again Shell) of the Debian Buster GNU Linux distribution that stops blinking after a few seconds : Does a program die? Surely not! If its Originator is Eternal and decides to keep rewriting it.

"By the ten nights! By the even! And by the odd!"

Qu'ran. Surah 89. Al Fajr (the Dawn). Verses 2 and 3.

Index

Bibliography

[And04] Yves André. *Une introduction aux motifs (motifs purs, mixtes, périodes)*, volume 17 of *Panoramas et Synthèses*. SMF, 2004. 62

[Ayo15] Joseph Ayoub. *Une version relative de la conjecture des périodes de Kontsevich-Zagier*. Annals of Mathematics, 3(181): p.905–992, 2015. 39

[Bak90] Alan Baker. *Transcendental number theory*. Cambridge University Press, London, 3th edition, 1990. 34

[Bri05] Martin Bright. *Brauer groups of diagonal quartic surfaces*. *Journal of symbolic computation*, 2005. preprint available at *http://www.boojum.org.uk/maths/articles/computing-br.pdf*. 21

[Bro12] Francis Brown. *Mixed Tate motives over* $\mathbb{Z}$. Annals of Mathematics, 2(175): p.949–976, 2012. 38

[BSD04] Martin Bright and Sir Peter Swinnerton-Dyer. *Computing the* Brauer-Manin *obstruction*. *Math. Proc. Camb. Phil. Soc.*, 137: p.1–16, 2004. 21

[CG66] John Cassels and Michael Guy. *On the Hasse principle for cubic surfaces*. *Mathematika*, (**13**): p.111–120, 1966. 13

[CMSP17] James Carlson, Stephan Müller-Stach, and Chris Peters. *Period Mappings and Period Domains*. Cambridge studies in advanced mathematics. Cambridge University Press, 2017. 60

[CTS21] Jean-Louis Colliot-Thélène and Alexei Skorobogatov. *The Brauer–Grothendieck Group*. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics. Springer International Publishing, 2021. 15

[CTSSD87a] Jean-Louis Colliot-Thélène, Jean-Jacques Sansuc, and Sir Peter Swinnerton-Dyer. *Intersection of two quadrics and Châtelet surfaces.* I. *J. reine angew. Math*, 373: p.37–107, 1987. 14

[CTSSD87b] Jean-Louis Colliot-Thélène, Jean-Jacques Sansuc, and Sir Peter Swinnerton-Dyer. *Intersection of two quadrics and Châtelet surfaces.* II. *J. reine angew. Math*, 374: p.72–168, 1987.

[Del80] Pierre Deligne. *Cycles de Hodge absolus et périodes des intégrales des variétés abéliennes.* Mémoires de la Société Mathématique de France, 2(2): p.23–33, 1980. Notes de J.L Brylinsky. 59

[DK76] Frank Demeyer and Max-Albert Knus. *The* Brauer *group of a real curve.* *Proc. Amer. Math. Soc.*, 57(2): p.233–270, 1976. 21

[EHKV01] Dan Edidin, Brendan Hassett, Andrew Kresch, and Angelo Vistoli. *Brauer groups and quotient stacks.* *Amer. J. Math.*, (**123**): p761–777, 2001. arXiv preprint AG99-05049 available at *https://arxiv.org/abs/math/9905049*. 14

[GKZ90] Israil.M Gelfand, Mikhail.M Kapranov, and Andrei.V Zelevinsky. Generalized Euler integrals and A-hypergeometric functions. *Advances in Mathematics*, 84(2):255–271, 1990. 39

[GR94] Izrail M. Gradshteyn and Iosif S. Ryzhik. *Table of integrals, series and products.* Academic Press, New York, 5th edition, 1994. 52

[Gro68] Alexander Grothendieck. *Le groupe de* Brauer I. II. III. In *Dix exposés sur la cohomologie des schémas.*, volume 3 of *Adv. Stud. Pure Math.*, chapter IV, V, VI., pages p.46–188. North Holland, Amsterdam, 1968. 15

[HFvWMS18] Annette Huber, Benjamin Friedrich, Jonas von Wangenheim, and Stephan Müller-Stach. *Periods and Nori Motives.* Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics. Springer International Publishing, 2018. 36, 48, 61

[KT04] Andrew Kresch and Yuri Tschinkel. *On the arithmetic of Del Pezzo surfaces of degree 2.* Proc. London Math. Soc., 89(3): p.545–569, 2004. 13

[KV03] Andrew Kresch and Angelo Vistoli. *On coverings of Deligne-Mumford stacks and surjectivity of the Brauer map.* arXiv preprint AG03-01249 available at *https://arxiv.org/abs/math/0301249*, 2003.

[Lan91] Serge Lang, editor. *Number theory III*, volume 60 of *Encyclopædia of Mathematical Sciences*. Springer Verlag, New York, 1991. 15

[Law89] Derek Lawden. *Elliptic functions and applications*, volume 80 of *Applied Mathematical Sciences*. Springer Verlag, Heidelberg, 1989. 45, 54

[Man74] Yuri Manin. *Cubic forms : algebra, geometry, arithmetic.* North Holland, Amsterdam, 1974. Translation of the original russian book by M. Hazewinkel. Юрий Манин. *Кубицеские формы : алгебра, геометря, арифметика.* Москва. Наука. 1972. 13, 16

[Mil68] Willard Miller. *Lie Theory and the Hypergeometric Functions.* Journal of Mathematics and Mechanics, 17(12): p.1143–1174, 1968. 38

[Mil70] James Milne. *The Brauer group of a rational surface.* Inv. Math., (11): p.304–307, 1970. 13

[Mil80] James Milne. *Étale cohomology.* Princeton University Press, 1980. 14

[Nik94] Viacheslav Nikulin. *On the* Brauer *group of real algebraic surfaces.* In *Algebraic Geometry and its applications*, volume 25 of *Aspects of Mathematics*, pages 113–136. Vieweg, 1994. 21

[NP96] Yuri Nesterenko and Patrice Philippon, editors. *Introduction to Algebraic Independence Theory*, volume 1752 of *Lecture Notes in Mathematics*. Springer Verlag, New York, 1996. 53, 57

[PBM86] Anatoli Prudnikov, Yuri Brychkov, and Oleg Marichev. *Elementary Functions*, volume 1 of *Integrals and Series*. Gordon and Breach Sci. Publ., New York, 1986. 56

[Pil91] Jonathan Pila. *Geometric postulation of a smooth function and the number of rational points. Duke Math. J.*, (63): p.449–463, 1991. 2

[PS98a] Alexei N. Parshin and Igor R. Shafarevich, editors. *Algebraic Geometry III. Complex Algebraic Varieties. Algebraic Curves and Their Jacobians*, volume 36 of *Encyclopædia of Mathematical Sciences*. Springer Verlag, New York, 1998.

[PS98b] Alexei N. Parshin and Igor R. Shafarevich, editors. *Number theory IV. Transcendental numbers*, volume 44 of *Encyclopædia of Mathematical Sciences*. Springer Verlag, New York, 1998. 34

[Shi80] Goro Shimura. *The periods of abelian varieties with complex multiplication and the spectral values of certain zeta functions.* Mémoires de la Société Mathématique de France, 2(2): p.103–106, 1980. 59

[Sko97] Alexei Skorobogatov. *Beyond the Manin obstruction.* Max-Planck-Institut für Mathematik. arXiv preprint 97-11006 available at *https://arxiv.org/abs/alg-geom/9711006*, 1997. 8

[Sko99] Alexei Skorobogatov. *Beyond the Manin obstruction.* Inv. Math., (135): p.399–424, 1999. 8

[Sko01] Alexei Skorobogatov. *Torsors and rational points*, volume 144 of *Cambridge Tracts in Mathematics.* Cambridge University Press, Cambridge, 2001. 15

[Vil68] Naum Jakoblevich Vilenkin. *Special Functions and the Theory of Group Representations*, volume 22 of *Translation of Mathematical Monographs.* American Mathematical Society, Providence, Rhode Island, 1968. 38

[Wal06] Michel Waldschmidt. *Transcendance des périodes : État des connaissances.* volume 1 of *Advances in Mathematics.* Tunisian Mathematical Society, African Diaspora Journal of Mathematics, Proceedings of the 11th symposium of the Tunisian Mathematical Society (15-18 march 2004) 2006. arXiv preprintNT05-02582 avalaible at *https://arxiv.org/abs/math.NT/0502582*. 62